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The Application of Substitutional Analysis to Invariants

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III—The Application of Substitutional Analysis to Invariants

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In a previous paper* it was proved that the generating function for any class of ternary concomitants might be obtained from the corresponding generating function for gradients (coefficient products) by multiplication by $(1 - x)(1 - y)(x - y)$. A generating function for ternary gradients was given in Theorem III of that paper, but it is of such a character that it is useless for purposes of calculation. In this paper a new system of generating functions is obtained applicable to perpetuants or to forms of finite order, and also to binary, ternary, or any forms.

In Section I a class of polynomial function $f_{\alpha}(z)$ is discussed which appeared in the paper just quoted in connection with the generating function for binary gradients of particular substitutional form in the perpetuant case. In Section II a one to one correspondence between binary perpetuants of particular substitutional form and the terms of the corresponding generating function is obtained by means of the tableau notation; and this is used to give a very simple extension of Grace's Theorem on irreducible perpetuants to the case of perpetuants of particular substitutional form. Section III deals with the properties of the functions for forms of finite order which correspond to the functions $f_{\alpha}(z)$ for perpetuants.

The generating functions for the different substitutional classes must by addition give the generating function for types. This in some cases has been obtained independently. Thus there arise certain algebraic identities. In Section IV a general theorem is established covering all these identities. It is obtained by means of the Characteristic Function of Schur. The same method is then used to express the binary generating functions in a new form.

In Section V, the Compound Symmetric group is discussed, which is formed by the simultaneous permutation of two sets of letters. It is shown that certain results due to Frobenius are applicable to the problem in hand. The results of this section are developed with a view to their application to the problem of ternary generating functions.

In Section VI, the functions $f_{\alpha}(x, y)$ are defined and discussed; they are the extension to two variables of the function $f_{\alpha}(z)$ of Section I, and they play a corresponding role in ternary generating functions. The theorems of Section (IV)

* Young, 'Proc. London Math. Soc.,' vol. 35, p. 425 (1933).

are then extended to these functions. The generating functions for ternary perpetuants thus obtained contain two sets of terms, only one of which is applicable to the problem in hand. The task of separating the function into two, one of which is the actual generating function required is carried out in some of the simpler cases.

In Section VII the generating function for ternary perpetuants is definitely established; and in the simplest cases it is compared to the corresponding symbolical products.

In Section VIII the generating function for ternary forms of finite order is obtained.

I—The Functions $f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$

§ 1. It has been proved* that the generating function for binary covariant types of degree δ, and substitutional form $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$ $(\Sigma\alpha = \delta)$, of quantics of order n is

$$(1 - z)\, z^{\varpi} D^{-h} (1 - z^n)^{h-1} (1 - z^{n-1})^{h-2} \ldots (1 - z^{n-h+2})$$
$$\cdot \prod_{r=1}^{h} (1 - z^{\alpha_r - r + h + 1}) \ldots (1 - z^{\alpha_r - r + n + 1}) \prod_{r<t} (1 - z^{\alpha_r - \alpha_t + t - r}),$$

where

$$D = (1 - z)(1 - z^2) \ldots (1 - z^n),$$

and

$$\varpi = (h - 1)\,\alpha_h + (h - 2)\,\alpha_{h-1} + \ldots + \alpha_2.$$

We will use the notation

$$[m] = 1 - z^m$$
$$[m]\,! = (1 - z)(1 - z^2) \ldots (1 - z^m)$$
$$[m]_r = (1 - z^m)(1 - z^{m-1}) \ldots (1 - z^{m-r+1}).$$

Then this generating function becomes

$$\text{(I)} \qquad z^{\varpi}\,[1] \prod_{r=1}^{h} \frac{[\alpha_r + n + 1 - r]_{\alpha_r}}{[\alpha_r + h - r]\,!} \cdot \prod_{r<t} [\alpha_r - \alpha_t + t - r].$$

When n is increased indefinitely, we obtain the perpetuant generating function

$$\text{(II)} \qquad z^{\varpi} \frac{[1] \prod_{r<t} [\alpha_r - \alpha_t + t - r]}{\prod [\alpha_r + h - r]\,!} = \frac{z^{\varpi}\,[1]}{[\delta]\,!}\,[f_{\alpha_1, \alpha_2, \ldots \alpha_h}],$$

the last symbol being introduced in formal analogy to the new notation, for there is a factor $[k]$ corresponding to each factor k in f. In the case of f_δ, the perpetuants of a single form, this generating function becomes the familiar function

$$\frac{[1]}{[\delta]\,!}.$$

It is more convenient to write $[f_{\alpha_1, \alpha_2, \ldots \alpha_h}]$ in the form $f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$.

* Young, 'Proc. London Math. Soc.,' vol. 35, p. 437 (1933).

§ 2. It follows from the substitutional identity

$$1 = \Sigma\, \mathrm{T}$$

that every covariant can be expressed as a sum of covariant types, each term of the sum being a type of some definite substitutional form $\mathrm{T}_{\alpha_1, \alpha_2, \dots \alpha_h}$. Hence the generating function for covariant types in general must be the sum of the generating functions for the types of each particular substitutional form. We apply this to binary perpetuant types and obtain

$$\text{(III)} \qquad \frac{1}{(1-z)^{s-1}} = \Sigma\, \frac{1}{[\delta]\,!}\, z^{\varpi} f_{\alpha_1, \alpha_2, \dots \alpha_h} \,.\, f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)\,;$$

for there will be f different covariants for each type of substitutional form T obtained by permutation of quantics. Thus for small values of δ,

$$(1-z)^{-1} = (1-z^2)^{-1}\{f_2 \,.\, f_2(z) + z f_{1^2} \,.\, f_{1^2}(z)\} = (1-z^2)^{-1}(1+z)$$

$$(1-z)^{-2} = \{(1-z^2)(1-z^3)\}^{-1}\{1 + 2z(1+z) + z^3\}$$

$$(1-z)^{-3} = \{(1-z^2)(1-z^3)(1-z^4)\}^{-1}\{1 + 3z(1+z+z^2) + 2z^2(1+z^2) + 3z^3(1+z+z^2) + z^6\}.$$

Equation (III) may be written

$$[\delta]\,!/[1]^{\delta} = \Sigma\, z^{\varpi} f_{\alpha_1, \alpha_2, \dots \alpha_h} \,.\, f_{\alpha_1, \alpha_2, \dots \alpha_h}(z) = (1+z)(1+z+z^2)\dots(1+z+z^2+\dots+z^{\delta-1}).$$

The sum of the coefficients in the expansion of the last expression is $\delta\,! = \Sigma f^2$; thus we are led to expect that $f(z)$ is always an integral function of z, and that the sum of its coefficients is f.

§ 3. The value of f was originally obtained by FROBENIUS* in the form

$$f_{\alpha_1, \alpha_2, \dots \alpha_h} = \delta\,! \left| \frac{1}{(\alpha_r - r + s)\,!} \right|,$$

(where r defines the row and s the column in the determinant) as an immediate deduction from the fact that $f_{\alpha_1, \alpha_2, \dots \alpha_h}$ is the coefficient of $x_1^{\alpha_1+h-1}\, x_2^{\alpha_2+h-2} \dots x_h^{\alpha_h}$, in

$$(x_1 + x_2 + \dots + x_h)^{\delta}\, \Delta,$$

where

$$\Delta = \{x_1\, x_2 \dots x_h\}'\, x_1^{h-1}\, x_2^{h-2} \dots x_{h-1},$$

is the alternant of the h variables x.

To make the next paragraph more clear, we deduce the ordinary form of f. Now

$$f \,.\, \Pi\, (\alpha_r + h - r)\,! = \phi\, \delta\,!,$$

where ϕ is integral.

* 'S. B. berl. math. Ges.,' p. 522 (1900).

We regard the determinant as a function of h unknown quantities α_r. Then ϕ is zero when

$$\alpha_r - r = \alpha_t - t.$$

Thus ϕ has a factor

$$\prod_{r<t} (\alpha_r - \alpha_t - r + t).$$

The total order of ϕ in the α's is $\binom{\delta}{2}$ showing that the remaining factor is numerical. Comparison of coefficients of suitable products of powers of the α's shows that this factor is unity. Hence we obtain the usual form

$$f = \delta\,!\,\frac{\prod_{r<t} (\alpha_r - \alpha_t - r + t)}{\prod (\alpha_r - r + h)\,!}.$$

§ 4. THEOREM I—*The function* $f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)$ *may be written as a determinant*

$$[\delta]\,!\left|\frac{1}{[\alpha_r - r + s]\,!}\right|,$$

where r *defines the row, and* s *the column of the element.*

Let Δ be the value of this product, then

$$\Delta \prod [\alpha_r + h - r]\,! = [\delta]\,!\,\phi,$$

where ϕ is an integral function of z.

Let $\psi(z)$ be an integral function of z with integral coefficients, then if

$$\psi(z) = 0,$$

when z is any root of $z^a = 1$, $\psi(z)$ has a factor $z^a - 1$. Now ϕ is zero when $\alpha_r - r = \alpha_t - t$, and $\alpha_r - r$, $\alpha_t - t$ only appear as indices of z, thus ϕ is zero whenever $z^{\alpha_r - r} = z^{\alpha_t - t}$, *i.e.*, ϕ has a factor

$$1 - z^{\alpha_r - \alpha_t + t - r}.$$

Thus ϕ is a function which contains all the factors of

$$\prod_{r<t} [\alpha_r - \alpha_t + t - r].$$

In view of the fact that $\alpha_1, \alpha_2, \dots \alpha_h$ may be looked on at present as arbitrary numbers, the only necessarily repeated factor in this is $(1 - z)^{\binom{h}{2}}$.

Now when z approaches the value 1, we may take $1 - z = \zeta$ a small quantity. Then when ζ is small

$$[k] = k\zeta\ ; \text{ and}$$

$$\Delta = \delta\,!\left|\frac{1}{(\alpha_r - r + s)\,!}\right| = f_{\alpha_1, \alpha_2, \dots \alpha_h}.$$

Thus Δ has no factor $1 - z$. The number of factors $1 - z$ in $\Pi\, [\alpha_r + h - r]\,!$ is

$$\Sigma\, (\alpha_r + h - r) = \delta + \binom{h}{2},$$

hence ϕ has the factor $(1 - z)^{\binom{h}{2}}$, and therefore the factor $\prod_{r>t} [\alpha_r - \alpha_t - r + t]$.

No other factor is possible for this is of the same order as ϕ, and the numerical factor to be attached is easily seen to be unity. Hence, according to definition

$$\Delta = f_{\alpha_1, \alpha_2, \dots \alpha_h}(z).$$

§ 5. Theorem II—*The function $f_{\alpha_1, \alpha_2, \dots \alpha}(z)$ is an integral function of z.*

The fact that $f_{\alpha_1, \alpha_2, \dots \alpha_h}$ is always an integer is not sufficient; for instance, the function $\frac{[4]\ [1]}{[2]\ [2]}$ is not integral.

We notice first that when $\Sigma\alpha = \delta$,

$$Q = \frac{[\delta]\,!}{[\alpha_1]\,!\,[\alpha_2]\,!\dots[\alpha_h]\,!}$$

is integral. For let

$$P_{r,\alpha} = [r+1]\,[r+2]\dots[r+\alpha],$$

then

$$\frac{P_{r,\alpha}}{[\alpha]\,!} = \frac{P_{r,\alpha-1}}{[\alpha-1]\,!} + z^{\alpha}\frac{P_{r-1,\alpha}}{[\alpha]\,!}.$$

Hence, noticing that $P_{2,1}/[1]$, and $P_{0,\alpha}/[\alpha]!$ are always integral, it is easy to see by induction that $P_{r,\alpha}/[\alpha]!$ is integral. It follows at once that Q is integral. Now the determinant form of $f(z)$ given in Theorem I is a sum of expressions such as Q, which are now seen to be integral, hence $f(z)$ is necessarily integral.

Corollary—*The sum of the coefficients of the powers of z in $f(z)$ is f.*

This follows at once since we proved in the last paragraph that $\lim_{z\to 1} \Delta = f$.

§ 6. Theorem III—*If the tableaux for the representations $T_{\alpha_1, \dots \alpha_h}$, $T_{\beta_1, \dots \beta_k}$, are obtained from each other by the change of rows into columns and vice versa*, i.e., *if they are conjugate representations, then*

$$f_{\alpha_1,\alpha_2 \dots \alpha_h}(z) = f_{\beta_1, \beta_2, \dots \beta_k}(z).$$

Consider the case $h = 3$,

$$f_{\alpha_1, \alpha_2, \alpha_3}(z) = [\delta]\,!\,\frac{[\alpha_1 - \alpha_2 + 1]\,[\alpha_1 - \alpha_3 + 2]\,[\alpha_2 - \alpha_3 + 1]}{[\alpha_1 + 2]\,!\,[\alpha_2 + 1]\,!\,[\alpha_3]\,!} = [\delta]\,!\,A/B.$$

Let

$$f_{\beta_1, \beta_2, \dots \beta_k}(z) = [\delta]\,!\,A'/B'.$$

It has to be shown that

$$AB' = A'B.$$

The values of β_r and $\beta_r + k - r$ are

$$\begin{array}{rcccccccc} r = & 1, & 2, \ldots & \alpha_3 & , \alpha_3 + 1 & , \ldots & \alpha_2 & , \alpha_2 + 1 & , \ldots \alpha_1, \\ \beta_r = & 3, & 3, \ldots & 3 & , \quad 2 & , \ldots & 2 & , \quad 1 & , \ldots 1, \end{array}$$
$$\beta_r + k - r = \alpha_1 + 2, \alpha_1 + 1, \ldots \alpha_1 - \alpha_3 + 3, \alpha_1 - \alpha_3 + 1, \ldots \alpha_1 - \alpha_2 + 2, \alpha_1 - \alpha_2, \ldots 1;$$

that is, $\beta_r + k - r$ has all the values from 1 up to $\alpha_1 + 2$, in turn excepting only $\alpha_1 - \alpha_2 + 1, \alpha_1 - \alpha_3 + 2$. In the general case it will have all the values from 1 up to $\alpha_1 + h - 1$, in turn with the exception of $\alpha_1 - \alpha_2 + 1$, $\alpha_1 - \alpha_3 + 2, \ldots \alpha_1 - \alpha_h + h - 1$, the numbers containing α_1, which appear in A.

Further, A′ is the product of those factors which are defined by the differences of the various values of $\beta_r + k - r$, there being one factor for each difference.

We insert now the missing numbers, and so obtain the complete series of numbers from 1 up to $\alpha_1 + 2$. The product of factors defined by these differences is

$$[\alpha_1 + 1]!\,[\alpha_1]!\,[\alpha_1 - 1]! \ldots [1]!,$$

which may be written $[\alpha_1 + 1]!!$.

Let P_1 be the product of those factors which are defined by the differences between $\alpha_1 - \alpha_2 + 1$ and the other numbers, P_2 the product defined by the differences between $\alpha_1 - \alpha_3 + 2$ and the other numbers, and P' the function $[\alpha_2 - \alpha_3 + 1]$ defined by the difference between these two numbers, there being only one such factor for $h = 3$.

Then

$$P'.\,[\alpha_1 + 1]!! = P_1 P_2 A',$$

where

$$P' = [\alpha_2 - \alpha_3 + 1],\ P_1 = [\alpha_1 - \alpha_2]!\,[\alpha_2 + 1]!,\ P_2 = [\alpha_1 - \alpha_3 + 1]!\,[\alpha_3]!.$$

Also

$$[\alpha_1 - \alpha_2 + 1]!\,[\alpha_1 - \alpha_3 + 2]!\,B' = [\alpha_1 + 2]!!.$$

Hence

$$AB' = \frac{[\alpha_1 + 2]!!\,[\alpha_2 - \alpha_3 + 1]}{[\alpha_1 - \alpha_2]!\,[\alpha_1 - \alpha_3 + 1]!} = A'B,$$

the required result. The method of proof is applicable to any value of h.

It will be seen from equation (II), § 1, that the highest power π of z in $f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$ is

$$\pi = \binom{\delta + 1}{2} + \sum_{r<t} (\alpha_r - \alpha_t + t - r) - \Sigma \binom{\alpha_r + h - r + 1}{2} = \binom{\delta}{2} - \Sigma \binom{\alpha_r}{2} - \varpi.$$

Now from the values for r and β_r given it is easy to see that

$$\varpi' = \sum_{r=1}^{k} (r - 1)\,\beta_r = \Sigma \binom{\alpha_r}{2},$$

hence

(IV) $$\pi + \varpi + \varpi' = \binom{\delta}{2}.$$

Consider then the series

$$(1-z)^{-\delta+1} = \frac{[1]}{[\delta]\,!}\{1 + \dots + z^{\varpi} f . f(z) + \dots + z^{\varpi'} f' . f'(z) + \dots + z^{\binom{\delta}{2}}\},$$

where f and f' are conjugate. Then write z^{-1} for z, and we have

$$z^{\delta-1}(1-z)^{-\delta+1} = \frac{z^{\binom{\delta+1}{2}-1}[1]}{[\delta]\,!}\{1 + \dots + z^{-\pi-\varpi} f . f(z) + \dots + z^{-\pi'-\varpi'} f' . f'(z) + \dots + z^{-\binom{\delta}{2}}\},$$

and in view of (IV) and the equality of $f(z)$ and $f'(z)$, we see that the generating functions for conjugate representations merely change places in the series after this transformation.

§ 7. The numbers f satisfy the relation*

$$(\delta+1) f_{\alpha_1, \alpha_2, \dots \alpha_h} = \sum_{r=1}^{h+1} \mathrm{E}_r f_{\alpha_1, \alpha_2, \dots \alpha_h},$$

where E_r is an operator which increases α_r by unity. This relation has its counterpart for the functions $f(z)$, as we proceed to show.

Theorem IV—*The functions $f(z)$ for types of degree δ are connected with those for types of degree $\delta + 1$, by the equations* :—

$$z^{\Sigma(s-1)\alpha s} f_{\alpha_1, \alpha_2, \dots \alpha}(z)\,[\delta+1] = \sum_{r=1}^{h+1} [1]\,\mathrm{E}_r\{z^{\Sigma(s-1)\alpha_s} f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)\}.$$

For

$$\frac{[1]\,\mathrm{E}_r\, z^{\Sigma(s-1)\alpha_s} f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)}{[\delta+1]\, z^{\Sigma(s-1)\alpha_s} f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)}$$

$$= z^{r-1}\frac{[1]}{[\alpha_r+1-r+h]}\prod_{s<r}\frac{[\alpha_s-\alpha_r-s+r-1]}{[\alpha_s-\alpha_r-s+r]}\prod_{s>r}\frac{[\alpha_r-\alpha_s-r+s+1]}{[\alpha_r-\alpha_s-r+s]}$$

$$= \left(\prod_{s\neq r}\frac{z^{\alpha_r+1+h-r}-z^{\alpha_s+h-s}}{z^{\alpha_r+h-r}-z^{\alpha_s+h-s}}\right)\frac{1-z^{-1}}{z^{\alpha_r+h-r}-z^{-1}}.$$

When $r = h+1$, we have $\displaystyle\prod_{s=1}^{h}\frac{1-z^{\alpha_s+h-s}}{z^{-1}-z^{\alpha_s+h-s}}$.

Consider the two functions

$$\phi(y) = y(y-z^{-1})\prod_{r=1}^{h}(y-z^{\alpha_s+h-s}),$$

whose roots are $y_1, y_2, \dots y_{h+2}$, where

$$y_r = z^{\alpha_r+h-r},\ (r<h+2)\ ;\ y_{h+2} = 0\ ;$$

and

$$\psi(y) = \prod_{s=1}^{h}(yz - z^{\alpha_s+h-s}),$$

* Young, 'Proc. London Math. Soc.,' vol. 28, p. 262 (1928).

whose order is less by 2 than that of $\phi(y)$. Then by a well-known theorem

$$\sum_{r=1}^{h+2} \frac{\psi(y_r)}{\phi'(y_r)} = 0.$$

Now

$$\frac{\psi(y_{h+2})}{\phi'(y_{h+2})} = -z, \quad \frac{\psi(y_{h+1})}{\phi'(y_{h+1})} = z \prod_{s=1}^{h} \frac{1 - z^{\alpha_s + h - s}}{z^{-1} - z^{\alpha_s + h - s}},$$

and

$$\frac{\psi(y_r)}{\phi'(y_r)} = \frac{1}{z^{\alpha_r + h - r}} \cdot \frac{z^{\alpha_r + 1 + h - r} - z^{\alpha_r + h - r}}{z^{\alpha_r + h - r} - z^{-1}} \prod_{s \neq r} \frac{z^{\alpha_r + 1 + h - r} - z^{\alpha_s + h - s}}{z^{\alpha_r + h - r} - z^{\alpha_s + h - s}}$$

$$= z \frac{[1]}{[\delta + 1]} \cdot \frac{E_r z^{\Sigma (s-1) \alpha_s} f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)}{z^{\Sigma (s-1) \alpha} f_{\alpha_1, \alpha_2, \dots \alpha_h}(z)}.$$

On putting in these values of the terms in the equation just obtained, and dividing by z, we obtain at once the result required.

One hoped to obtain an analogue to the equation*

$$f_{\alpha_1, \alpha_2, \dots \alpha_h} = \Sigma\, \delta_r f_{\alpha_1, \alpha_2, \dots \alpha_h},$$

where δ_r indicates the decrease of α_r by unity; but the attempt failed. The nearest approach obtained to such an analogue was

$$f_{\alpha, 2}(z) = z f_{\alpha-1, 2}(z) + z^2 f_{\alpha, 1}(z^2),$$

but this relation is accidental rather than general.

II—Binary Perpetuants

§ 8. A one to one correspondence between terms in the generating function with the covariants themselves, for the case of perpetuants, can be obtained as follows. In the first place, when the substitutional form is T_δ, the covariants are covariants of a single form. It has been proved (Q.S.A. VII)† that all such forms can be expressed linearly in terms of the forms

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-2}\, a_{\delta-1})^{\lambda_{\delta-2}} (a_{\delta-1} a_\delta)^{2\lambda_{\delta-1}},$$

where

$$\lambda_{\delta-1} \geqq \lambda_{\delta-2} \geqq \lambda_{\delta-3} \dots \geqq \lambda_1;$$

of which the first term sequence C is

$$A_{\lambda_1} A_{\lambda_2} \dots A_{\lambda_{\delta-2}} A^2_{\lambda_{\delta-1}}.$$

The generating function for T_δ is

$$\frac{1}{(1-z^2)(1-z^3)\dots(1-z^\delta)} = \Sigma\, z^{\Sigma r \mu_r},$$

where r is one of the numbers $2, 3, \dots \delta$.

* Young, 'Proc. London Math. Soc.,' vol. 28, p. 261 (1928).
† Young, 'Proc. London Math. Soc.,' vol. 36, p. 339 (1934).

The correspondence is established by means of a diagram of rows and columns of dots, all rows begin with the same column on the left, and all columns with the same upper row like the tableaux. We represent $\delta\mu_\delta$ first by μ_δ columns of δ dots, then immediately to the right we put $\mu_{\delta-1}$ columns of $\delta - 1$ dots and so on, finally we have on the extreme right μ_2 columns of two dots. Thus this is a perfectly general diagram of dots except that there are no columns with only one dot, and there are no columns with more than δ dots. Then there are δ rows, and calling the rows beginning with the first $\lambda_\delta, \lambda_{\delta-1} \ldots \lambda_1$, we have

$$\lambda_\delta = \lambda_{\delta-1} \geqq \lambda_{\delta-2} \geqq \ldots \geqq \lambda_1.$$

Thus the symbolical product representation, or the leading gradients sequence C, is made to correspond to the generating function terms by means of the rows and columns of a diagram.

§ 9. Let PN belong to $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$, then any covariant belonging to this representation may be written in the form PNC. This will be looked on as a symbolical product. Now a symbolical letter a appears in two kinds, a_1, a_2, the letters of the first kind will be replaced here by 1, so that the symbolical letters appear only in one kind, and the symbolical product is not homogeneous. Let $T_\delta D$ be a symbolical product sum representing a covariant of a single form. Then

$$\text{PNC} \, . \, T_\delta D$$

is also a symbolical product sum, and it is of the substitutional form $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$. Moreover, the leading gradient of this new form is given at once by the product of the symbolical expressions of the leading gradients of its two component parts. Thus, when we have obtained covariants of this substitutional form and of weights given by the generating function $z^\varpi f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$, $f_{\alpha_1, \alpha_2, \ldots \alpha_h}$ in number, we can write down the rest corresponding to the full generating function

$$z^\varpi f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z) \; [1]/[\delta] \; !$$

by multiplication of symbolical forms; or by addition of suffixes in leading gradients; or by addition of indices in the symbolical form given in the last paragraph.

§ 10. Any symbolical product representing a perpetuant type of degree δ can be expressed in terms of the products

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

with a pre-selected letter a_1 in each symbolical factor. And the generating function $(1 - x)^{-\delta+1}$ simply represents all possible types of this form, they are linearly independent. By the use of the symbolical identities and STROH's Lemma all such forms can be expressed in terms of forms obtainable by permutation from

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-2} a_{\delta-1})^{\lambda_{\delta-2}} (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}+\lambda_\delta},$$

where

(V) $$\lambda_{\delta-1} \geqq \lambda_{\delta-2} \geqq \ldots \geqq \lambda_1,$$

and $\lambda_\delta - \lambda_{\delta-1} = 0$ or 1. The leading gradient sequence C of this form is

$$A_{1,\lambda_1} A_{2,\lambda_2} \ldots A_{\delta-1,\lambda_{\delta-1}} A_{\delta,\lambda_\delta} \equiv X,$$

where the first suffix defines the quantic and the second the weight.

Then all possible leading gradients can be expressed in terms of the gradients X, and those obtained by permutation of quantics from X.

If we operate on X with $\{A_{r_1} A_{r_2} \ldots A_{r_\rho}\}'$, the result is zero unless $\lambda_{r_1}, \lambda_{r_2} \ldots \lambda_{r_\rho}$ are all different. Consider the operation of $P_\rho N_\rho$ on X, where $P_\rho N_\rho$ is derived from a tableau F_ρ of $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$. Let Λ_ρ be the tableau obtained from F_ρ by replacing each letter A_r by its corresponding weight suffix λ_r in X; then $P_\rho N_\rho X$ is zero unless the numbers in every column of Λ_ρ are all different. We shall call Λ_ρ the weight tableau of X.

Definition—*A normal leading gradient* X_ρ *corresponding to a standard tableau* F_ρ *is one in which the weights satisfy the conditions* (V) *the quantics being arranged in proper sequence, and for which the weight tableau* Λ_ρ *is such that* $\lambda_{r+1} > \lambda_r$ *when* λ_{r+1} *is on a lower row than* λ_r. *And also* $\lambda_\delta = \lambda_{\delta-1}$ *when* λ_δ *does not lie on a lower row than* $\lambda_{\delta-1}$, *and* $\lambda_\delta = \lambda_{\delta-1} + 1$ *when* λ_δ *lies on a lower row.*

For a normal leading gradient X_ρ it is obvious at once that $P_\rho N_\rho X_\rho$ is not zero.

THEOREM V—*For any leading gradient* X, *and any standard tableau* F_ρ, *either*

(i) $P_\rho N_\rho X = 0,$

or

(ii) X *is a normal leading gradient corresponding to* F_ρ,

or

(iii) $P_\rho N_\rho X = \Sigma\, \beta s P_\sigma N_\sigma Y_\sigma,$

where Y_σ *is a normal leading gradient corresponding to the standard tableau* F_σ *from which* Y_σ *is derived; s is some permutation and* β *is numerical.*

The sequence of quantics in X will be first supposed to be the fixed pre-arranged sequence, so that according to conditions (V) $\lambda_{r+1} \geqq \lambda_r$ always. Consider the weights in order commencing with λ_1, if $\lambda_2 > \lambda_1$, $\lambda_3 > \lambda_2$ and so on, the conditions that X may be normal are being fulfilled. Let the sign of equality first appear in the case $\lambda_{r+1} = \lambda_r$, then unless A_{r+1} lies on a lower row than A_r, X is still normal for F_ρ. Let us then suppose that A_{r+1} lies in a lower row than A_r, but not in the same column, otherwise

$$P_\rho N_\rho X = 0.$$

Then, since

$$\lambda_{r+1} = \lambda_r, \qquad X = (A_r A_{r+1})\, X,$$

and

$$P_\rho N_\rho X = (A_r A_{r+1})\, P_\sigma N_\sigma X,$$

where F_σ is a standard tableau derived from F_ρ by the interchange of A_r and A_{r+1}. Now A_{r+1} lies in a higher row than A_r in F_σ and thus X is normal for F_σ up to this point. Proceeding thus step by step we find a standard tableau F_σ which is such that

$$P_\rho N_\rho X = s P_\sigma N_\sigma X,$$

where s is some permutation, and X is a normal leading gradient for F_σ at least so far as the letter $A_{\delta-1}$ is concerned; and, indeed, as far as the last letter is concerned, unless A_δ does not lie in a lower row than $A_{\delta-1}$ and $\lambda_\delta = \lambda_{\delta-1} + 1$.

In this case the last condition of the definition is not fulfilled. For this case it is necessary to consider the symbolical form Z from which the leading gradient was obtained.

Since $\lambda_\delta + \lambda_{\delta-1}$ is odd

$$\{A_{\delta-1}A_\delta\}\, Z = \Sigma\, Z',$$

where Z' is a form which has an increased value of $\lambda_\delta + \lambda_{\delta-1}$; the theorem will be assumed true for such forms when the total weight does not exceed that of X. Then

$$P_\sigma N_\sigma X = -\, P_\sigma N_\sigma\, (A_{\delta-1}A_\delta)\, X + \Sigma\, \beta s P_\tau N_\tau Y_\tau$$
$$= -\, (A_{\delta-1}A_\delta)\, P_\varpi N_\varpi X + \Sigma\, \beta s P_\tau N_\tau Y_\tau,$$

here F_ϖ is derived from F_σ by the interchange of $A_{\delta-1}$ and A_δ. Thus, $A_{\delta-1}$ is raised in the tableau in changing from F_σ to F_ϖ, and X is normal for F_ϖ.

The theorem is thus true when the quantics in X are arranged according to the fixed sequence; when this is not the case

$$X = sY,$$

where the quantics in Y are arranged in the fixed sequence.

Then

$$P_\rho N_\rho X = P_\rho N_\rho s Y = \Sigma \beta s' P_\sigma N_\sigma Y,$$

from the former case. Thus the theorem is always true.

§ 11. THEOREM VI.—*Every leading gradient can be linearly expressed in terms of forms obtained by permutation from* $P_\rho N_\rho X_\rho$; *where* X_ρ *is a normal leading gradient for the tableau* F_ρ *from which* $P_\rho N_\rho$ *is derived.*

Let X be a leading gradient; then

$$X = \Sigma\, TX = \Sigma\, \Sigma\, PNMX = \Sigma\, \beta s P_\rho N_\rho X_\rho,$$

by the last theorem.

The number of different forms obtainable from $P_\rho N_\rho X_\rho$ by permutation, *i.e.*, by left-hand multiplication with a permutation, is $f_{\alpha_1, \alpha_2, \dots \alpha_h}$, when the tableau F_ρ belongs to $T_{\alpha_1, \alpha_2, \dots \alpha_h}$.

Amongst the normal leading gradients corresponding to F_ρ, there is one which is of special importance, the normal leading gradient of minimum weight. This is obtained by the rule that when A_{r+1} is in a lower row of F_ρ than A_r

$$\lambda_{r+1} = \lambda_r + 1, \quad \text{and otherwise} \quad \lambda_{r+1} = \lambda_r, \quad \text{while} \quad \lambda_1 = 0.$$

For example, the standard tableaux of T_{32} are

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & \end{pmatrix}, \quad \begin{pmatrix} A_1 & A_2 & A_4 \\ A_3 & A_5 & \end{pmatrix}, \quad \begin{pmatrix} A_1 & A_2 & A_5 \\ A_3 & A_4 & \end{pmatrix}, \quad \begin{pmatrix} A_1 & A_3 & A_4 \\ A_2 & A_5 & \end{pmatrix}, \quad \begin{pmatrix} A_1 & A_3 & A_5 \\ A_2 & A_4 & \end{pmatrix};$$

the corresponding weight tableaux for normal leading gradients of minimum weight are

$$\begin{pmatrix}0 & 0 & 0\\ 1 & 1 & \end{pmatrix}, \quad \begin{pmatrix}0 & 0 & 1\\ 1 & 2 & \end{pmatrix}, \quad \begin{pmatrix}0 & 0 & 1\\ 1 & 1 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 1\\ 1 & 2 & \end{pmatrix}, \quad \begin{pmatrix}0 & 1 & 2\\ 1 & 2 & \end{pmatrix}.$$

It will be seen at once that these correspond to the terms of the generating function $z^2 f_{3,2}(z)$. It will be found that in every case the normal leading gradients of minimum weight have $z^{\varpi} f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$ for generating function.

§ 12. Let

$$X = \prod_{r=1}^{\delta} A_{r, \lambda_r}, \quad Y = \prod_{r=1}^{\delta} A_{r, \mu_r},$$

be both normal leading gradients corresponding to the standard tableau F, Y being the minimum leading gradient.

Then $\lambda_{r+1} - \lambda_r \geqq \mu_{r+1} - \mu_r$, and therefore $\lambda_{r+1} - \mu_{r+1} \geqq \lambda_r - \mu_r$; while $\lambda_\delta - \mu_\delta = \lambda_{\delta-1} - \mu_{\delta-1}$. Let $\lambda_r - \mu_r = v_r$. Then $Z = \prod_{r=1}^{\delta} A_{r, v_r}$ is a normal leading gradient corresponding to the tableau for T_δ which has only one row. Moreover, $T_\delta Z_\delta$ is the general leading gradient for a single quantic; and these forms are enumerated by the generating function $[1]\{[\delta]\,!\}^{-1}$ as described in §8.

Thus the normal leading gradients for $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$ are enumerated by the generating function $g\,[1]\{[\delta]\,!\}^{-1}$, where g is the generating function for the minimum normal leading gradient. But this we have found to be (§9)

$$z^{\varpi} f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)\;[1]\{[\delta]\,!\}^{-1},$$

hence

$$g = z^{\varpi} f_{\alpha_1, \alpha_2, \ldots \alpha_h}.$$

THEOREM VII—*The generating function for the minimum normal leading gradients corresponding to* $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$ *is*

$$z^{\varpi} f_{\alpha_1, \alpha_2, \ldots \alpha_h}[z].$$

It is easy to see that the lowest weight of the minimum normal leading gradients corresponding to $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$ is given by the first tableau, that in which the first α_1 letters lie in the first row, the next α_2 in the second row and so on. In the corresponding weight tableau every element of the r^{th} row is $r - 1$, and the total weight is

$$\alpha_2 + 2\alpha_3 + \ldots + (h-1)\,\alpha_h = \varpi.$$

Thus the leading gradients are all identified in correspondence with the terms of the generating function.

§ 13. There is no difficulty in extending these results to irreducible perpetuants. GRACE* proved that all perpetuant types can be expressed in terms of those of the form

$$\text{(VI)} \qquad (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-2} a_{\delta-1})^{\lambda_{\delta-2}} (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}},$$

* 'Proc. London Math. Soc.,' vol. 35, p. 107 (1902); and GRACE and YOUNG, "Algebra of Invariants," pp. 327-330 and Appendix IV (1903).

where

$$\lambda_1 \geqq 1, \qquad \lambda_2 \geqq 2, \ldots \qquad \lambda_{\delta-2} \geqq 2^{\delta-3}, \qquad \lambda_{\delta-1} \geqq 2^{\delta-2},$$

and the sequence of letters is fixed; and of products of forms. The generating function for the forms (VI) is $\dfrac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}$.

Wood* demonstrated that this result is exact, or, in other words, that there can be no linear relation between the forms (VI) and products of forms.

Grace† proved further that for a single form where the letters are interchangeable the index conditions for the forms (VI) may be written

$$\lambda_1 \geqq 1, \quad \lambda_2 \geqq 1 + \lambda_1, \quad \lambda_3 \geqq 2 + \lambda_2, \ldots \quad \lambda_{\delta-2} \geqq 2^{\delta-4} + \lambda_{\delta-3}, \quad \lambda_{\delta-1} \geqq 2\lambda_{\delta-2};$$

the generating function being

$$\frac{x^{2^{\delta-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\delta)}.$$

The argument used establishes at once that all perpetuant types may be expressed linearly in terms of products, and of forms

$$(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_{\delta-2}a_{\delta-1})^{\lambda_{\delta-2}}(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}+\lambda_\delta},$$

where the quantics are arranged in fixed sequence, and

$$\lambda_1 \geqq 1, \qquad \lambda_2 \geqq 1 + \lambda_1, \ldots \qquad \lambda_{\delta-2} \geqq 2^{\delta-4} + \lambda_{\delta-3}, \qquad \lambda_{\delta-1} \geqq 2^{\delta-3} + \lambda_{\delta-2}$$

and $\lambda_\delta - \lambda_{\delta-1} = 0$ or 1; and of forms obtained from these by permutation. The leading gradient sequence C of this form is

$$A_{1,\lambda_1}A_{2,\lambda_2}\ldots A_{\delta,\lambda_\delta}.$$

The whole of the preceding argument may be applied to this result, the generating function for perpetuant types being multiplied in every case by $x^{2^{\delta-1}-1}$ to give the corresponding generating function for irreducible perpetuant types. The normal leading gradient for an irreducible perpetuant corresponding to a given tableau is the same in form as that in the former case except that each weight suffix λ_r is increased by 2^{r-1}, except in the case of λ_δ which is increased by $2^{\delta-2}$.

III—Application to Binary Forms of Finite Order

§ 14. The generating function for covariant types of substitutional form $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$ of the binary n—*ic* has been given in (I), § 1. This will be written

$$z^\varpi f^n_{\alpha_1, \alpha_2, \ldots \alpha_h}(z);$$

* 'Proc. London Math. Soc.,' vol. 1, p. 480 (1904).

† 'Proc. London Math. Soc.,' vol. 35, p. 319 (1903) and Grace and Young, *op. cit.*

it is an integral function of z, for it was obtained in the first place in the paper quoted in § 1 in the form

$$(1-z)\left|\phi\binom{\alpha_r+n-r+s}{n}\right|,$$

in which all the elements of the determinant are integral functions of z; hence the function itself is an integral function of z. Comparison with $f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$ gives the relation

(VII) $$[\delta]!\, f^n_{\alpha_1, \alpha_2, \ldots \alpha_h}(z) = [1]\, f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z) \prod_{r=1}^{h} [\alpha_r + n + 1 - r]_{\alpha_r}.$$

The highest index of z in $f^n_{\alpha_1, \alpha_2, \ldots \alpha_h}(z)$ must then be

$$1 + \pi + \Sigma\binom{\alpha_r+1}{2} + \Sigma\,\alpha_r\,(n+1-r) - \binom{\delta+1}{2} = 1 + \delta n - 2\varpi = \theta,$$

on using the notation and results of § 6.

Replace z by z^{-1} in (VII) and multiply the result by $z^{\binom{\delta+1}{2}+\theta}$ the right-hand side is simply multiplied by $(-)^{\delta+1}$, hence

$$f^n_{\alpha_1, \alpha_2, \ldots \alpha_h}(z) = \Sigma\, B_r\,(z^r - z^{\theta-r}).$$

From the generating function for covariants linear in the coefficients of each of δ quantics of order n, we obtain—in the same way as for perpetuants—

$$\frac{[n+1]^\delta}{[1]^{\delta-1}} = \Sigma\, f_{\alpha_1, \alpha_2, \ldots \alpha_h}\, z^{\varpi} f^n_{\alpha_1, \alpha_2, \ldots \alpha_h}(z).$$

When $T_{\alpha_1, \alpha_2, \ldots \alpha_h}$, $T_{\beta_1, \beta_2, \ldots \beta_k}$ are conjugate, and their tableaux differ by interchange of rows and columns, the generating functions are not quite so simply related as in the case of perpetuants. There is a relation, as we proceed to show. As in § 6 we take $h = 3$, and use the value of β_r as found there. Then

$$\begin{aligned}\Pi\,[\beta_r + n + 1 - r]_{\beta_r} &= [n+3]_3\,[n+2]_3 \ldots [n+4-\alpha_3]_3 \\ &\quad [n+2-\alpha_3]_2\,[n+1-\alpha_3]_2 \ldots [n+3-\alpha_2]_2 \\ &\quad [n+1-\alpha_2]\,[n-\alpha_2] \ldots [n+2-\alpha_1] \\ &= [n+3]_{\alpha_3}\,[n+2]_{\alpha_2}\,[n+1]_{\alpha_1}.\end{aligned}$$

Now

$$\begin{aligned}[n+r]_{\alpha_r} &= \prod_{s=1}^{\alpha_r} [n + r - \alpha_r + s] \\ &= (-)^{\alpha_r} z^{-(m+1-r)\,\alpha_r - \binom{\alpha_r+1}{2}}\,[m + 1 + \alpha_r - r]_{\alpha_r},\end{aligned}$$

where

$$n = -m - 2.$$

Hence

$$\Pi\,[n+r]_{\alpha_r} = (-)^\delta z^\chi\, \Pi\,[m + 1 + \alpha_r - r]_{\alpha_r},$$

where

$$\chi = -(m+1)\,\delta + \varpi - \varpi'.$$

The other part of $f^n_{\alpha_1, \alpha_2, \dots \alpha_h}(z)$ does not contain n, and is the same as for the conjugate set as was seen in § 6. Thus the transformation of n into $-m-2$ which changes $\frac{[n+1]^\delta}{[1]^{\delta-1}}$ into $(-)^\delta z^{-(m+1)\delta} \cdot \frac{[m+1]^\delta}{[1]^{\delta-1}}$ at the same time interchanges conjugate generating functions.

IV—The Characteristic Function

§ 15. In the fourth paper on Quantitative Substitutional Analysis* the characteristic function $\Phi_{\alpha_1, \alpha_2, \dots \alpha_h}$ of the symmetric group corresponding to the representation $T_{\alpha_1, \alpha_2, \dots \alpha_h}$ was introduced. The conception is due to Schur. The characteristic function is defined as the function

$$\Phi_{\alpha_1, \alpha_2, \dots \alpha_h} = \Sigma \frac{\chi_{\beta_1, \beta_2, \dots \beta_n}}{\beta_1 ! \ \beta_2 ! \dots \beta_n !} \left(\frac{s_1}{1}\right)^{\beta_1} \left(\frac{s_2}{2}\right)^{\beta_2} \dots \left(\frac{s_n}{n}\right)^{\beta_n},$$

where $\chi_{\beta_1, \beta_2, \dots \beta_n}$ is the characteristic of the permutation which has β_r cycles of r letters each, for the various values of r. It was shown in the paper referred to that when every cycle of degree r is replaced by the symbol s_r then $T_{\alpha_1, \alpha_2, \dots \alpha_h}$ becomes $f_{\alpha_1, \alpha_2, \dots \alpha_h} \Phi_{\alpha_1, \alpha_2, \dots \alpha_h}$, $P_r N_r M_r$ becomes $\Phi_{\alpha_1, \alpha_2, \dots \alpha_h}$ and $P_r \sigma_{rs} N_s M_s$ becomes zero when r and s are different.

Theorem VIII—*The characteristic function* $\Phi_{\alpha_1, \alpha_2, \dots \alpha_h}$ *is equal to* $|\Phi_{\alpha_r+s-r}|$, *where* r, s *define the row and column respectively in the determinant; and* Φ_λ *is the characteristic function of the representation* T_λ *of the symmetric group of* λ *letters.*

This result follows at once from the equation proved in Q.S.A. VI†

$$\text{(VIII)} \qquad \frac{n!}{f_{\alpha_1, \alpha_2, \dots \alpha_h}} T_{\alpha_1, \alpha_2, \dots \alpha_h} = \prod_{r<s} (1 - S_{rs}) \frac{1}{n!} \binom{n}{\alpha_1 \alpha_2 \dots \alpha_h} \Gamma P_{\alpha_1, \alpha_2, \dots \alpha_h},$$

where $P_{\alpha_1, \alpha_2, \dots \alpha_h}$ is the product of symmetric groups of degrees $\alpha_1, \alpha_2, \dots \alpha_h$ respectively; Γ indicates that the sum of all the $n!$ permutations of the n letters are to be taken, S_{rs} represents the operation of moving one letter from the s^{th} row of the tableau up to the r^{th} row, and the resulting term is taken to be zero when any row becomes less than a row below it or when letters from the same row overlap. In fact, when cycles of degree r are replaced by the symbol s_r this equation becomes

$$\Phi_{\alpha_1, \alpha_2, \dots \alpha_h} = \prod_{r<s} (1 - S_{r,s}) \Phi_{\alpha_1} \Phi_{\alpha_2} \dots \Phi_{\alpha_h},$$

which gives the result stated, provided the restriction as to zero terms is observed. It will be noticed that the suffix of the element Φ_{α_r+s-r} in the determinant may become zero or negative. In the former case the value of the element is unity, and in the latter case it is zero.

* Young, 'Proc. London Math. Soc.,' vol. 31, p. 259 (1930).

† Young, 'Proc. London Math. Soc.,' vol. 34, p. 199 (1932).

COROLLARY—*The coefficient in* $\Phi_{\alpha_1, \alpha_2, \dots \alpha_h}$ *of any permutation which contains a cycle of degree greater than* $\alpha_1 + h - 1$ *is zero. And hence the characteristic of such a permutation is zero.*

We may use equation (VIII) to put Theorem VIII in another useful form. The symbol G_α is used for the positive symmetric group of α letters divided by its order, then

THEOREM IX—

$$T_{\alpha_1, \alpha_2, \dots \alpha_h} = \frac{f_{\alpha_1, \alpha_2, \dots \alpha_h}}{n\,!} \Gamma \mid G_{\alpha_r + s - r} \mid,$$

where r, s define the rows and columns of the determinant; and in its expansion every letter appears in one and only one factor of each term.

§ 16. From the equation

$$1 = \Sigma\, T_{\alpha_1, \alpha_2, \dots \alpha_h},$$

we deduce at once the equation

$$1 = \Sigma\, f_{\alpha_1, \alpha_2, \dots \alpha_h}\, \Phi_{\alpha_1, \alpha_2, \dots \alpha_h},$$

and hence by Theorem VIII the equation

$$1 = \Sigma\, f_{\alpha_1, \alpha_2, \dots \alpha_h} \mid \Phi_{\alpha_r + s - r} \mid.$$

Now

$$\Phi_1,\ \Phi_2,\ \dots\ \Phi_n$$

are a series of functions each of which contains a new independent variable which does not appear in those which precede it; thus s_r first appears in Φ_r. The functions may then be regarded as completely arbitrary and independent. Hence

THEOREM X—*The quantities* Φ_r *for positive integral values of r are arbitrary, for negative values of r they are zero, and* $\Phi_0 = 1$, *then*

$$\Phi_1^{\,n} = \sum_{\Sigma \alpha = n} f_{\alpha_1, \alpha_2, \dots \alpha_h} \mid \Phi_{\alpha_r + s - r} \mid.$$

That this equation is true when Φ is a generating function is an immediate deduction from Theorem VIII in Some Generating Functions,* and hence it is true for a wide range of functions. It was well to determine, whether it expressed an inherent property of generating functions as such, we see that it does not, that, in fact, it is not a property of Φ at all, but only of the numbers $f_{\alpha_1, \alpha_2, \dots \alpha_h}$. That the value of the left-hand side of the equation is $\Phi_1^{\,n}$ is seen by considering the last term of the series

$$f_{1^n} \mid \Phi_{1+s-r} \mid,$$

which alone of all the terms contains $\Phi_1^{\,n}$, the only term on the right-hand side not containing one of the other Φ's.

§ 17. LEMMA I.

$$\sum_{r=1}^{\delta} \frac{1}{[r]\,.\,[\delta - r]\,!} = \frac{\delta}{[\delta]\,!}.$$

* YOUNG, 'Proc. London Math. Soc.,' vol. 35, p. 440 (1933).

It is quite easy to verify this for small values of δ, so its truth will be assumed for all values of δ less than that one under consideration.

Multiply each side by $[\delta]!$ then it is required to prove that

$$\phi_\delta(z) = \sum_{r=1}^{\delta} \frac{[\delta]_r}{[r]} - \delta = 0.$$

Then

$$\begin{aligned}\phi_\delta(z) - \phi_{\delta-1}(z) &= z^{\delta-1} + [\delta-1]\,z^{\delta-2} + [\delta-1]_2\,z^{\delta-3} + \ldots + [\delta-1]_{\delta-2}\,z + [\delta-1]\,! - 1 \\ &= [\delta-1]\{z^{\delta-2} + [\delta-2]\,z^{\delta-3} + [\delta-2]_2\,z^{\delta-4} + \ldots + [\delta-2]\,! - 1\} \\ &= [\delta-1]\,\{\phi_{\delta-1}(z) - \phi_{\delta-2}(z)\}.\end{aligned}$$

The truth of the lemma is then at once apparent by induction.

§ 18. THEOREM XI—*When in the generalized form* Φ_δ *of the symmetric group the symbol* s_r *for a cycle of degree* r *is replaced by* $(1 - z^r)^{-1}$, *the replacement being made for all cycles*,

$$\Phi_\delta = \frac{\delta\,!}{[\delta]\,!}.$$

This is verified without difficulty for $\delta = 1, 2, 3$; it will be assumed true for all degrees less than δ. In the symmetric group we have first those terms in which the last letter b is not permuted, these form the symmetric group of degree $\delta - 1$, then those terms in which b occurs in a cycle of two letters and so on. Let P_r denote the sum of all terms in which b occurs in a cycle of r letters. Then the sum of all terms is $\sum_{r=1}^{\delta} P_r$.

Consider P_r when the $r - 1$ letters associated with b are fixed, we may have the remaining $\delta - r$ letters permuted by any permutation of the symmetric group of degree $\delta - r$. Also the $r - 1$ letters may be selected in $\binom{\delta-1}{r-1}$ ways and also may have $(r - 1)\,!$ different positions in the cycle relative to b. Then when the replacement of cycles is made

$$P_r = \frac{(\delta-1)\,!}{[r]\,[\delta-r]\,!}.$$

And

$$\Phi_\delta = \sum_{r=1}^{\delta} \frac{(\delta-1)\,!}{[r]\,[\delta-r]\,!} = \frac{\delta\,!}{[\delta]\,!},$$

by the lemma, which proves the theorem.

§ 19. THEOREM XII—

$$\delta\,!\,z^{\varpi_\alpha} f_\alpha(z) = \sum_\beta h_\beta\,\chi_\beta^{(\alpha)} \frac{[\delta]\,!}{[\beta_1]\,[\beta_2]\ldots[\beta_k]}.$$

Here α, β are generic symbols, and β stands for $\beta_1, \beta_2, \ldots \beta_k$, where

$$\beta_1 + \beta_2 + \ldots + \beta_k = \delta\,;$$

and h_β is the number of members of the β conjugate class of the symmetric group which consists of operations having cycles of degrees $\beta_1, \beta_2, \ldots \beta_k$; $\chi_\beta^{(\alpha)}$ is the characteristic of this class in the representation T_α.

By Theorem VIII the characteristic function for this representation is

$$\Phi_\alpha \equiv \Phi_{\alpha_1, \alpha_2, \ldots \alpha_h} = | \Phi_{\alpha_r + s - r} |.$$

Here the cycles are represented by variables $s_1, s_2, \ldots s_\delta$, and the characteristic function is at once given by this equation with the ordinary rules of algebra. If we write $(1 - x^r)^{-1}$ for s_r we obtain

$$\Phi_{\alpha_r + s - r} = \frac{1}{[\alpha_r + s - r]\,!}.$$

But by Theorem I, § 4,

$$z^{\varpi} f_{\alpha_1, \alpha_2, \ldots \alpha_h}(z) \equiv [\delta]\,! \left| \frac{1}{[\alpha_r + s - r]\,!} \right| = [\delta]\,!\,\Phi_\alpha = \frac{1}{\delta\,!} [\delta]\,! \sum_\beta h_\beta\, \chi_\beta^{(\alpha)} s_{\beta_1} s_{\beta_2} \ldots s_{\beta_k}$$

$$= \frac{1}{\delta\,!} \Sigma\, h_\beta\, \chi_\beta^{(\alpha)} \frac{[\delta]\,!}{[\beta_1]\,[\beta_2] \ldots [\beta_k]};$$

which was to be proved.

THEOREM XIII—

$$\frac{[\delta]\,!}{[\beta_1]\,[\beta_2] \ldots [\beta_k]} = \sum_\alpha \chi_\beta^{(\alpha)} z^{\varpi_\alpha} f_\alpha(z).$$

To prove this we use the well-known results

$$\sum_\alpha \chi_\gamma^{(\alpha)} \chi_\beta^{(\alpha)} = 0, \qquad \beta \neq \gamma; \qquad \sum_\alpha (\chi_\beta^{(\alpha)})^2 = \delta\,!/h_\beta.$$

Multiply the identity of the last theorem by $\chi_\gamma^{(\alpha)}$ and then sum the result for all representations α, we obtain the desired result at once.

Equation (III), § 2, is a particular case of this theorem.

§ 20. The generating functions for binary forms of finite order may be treated in the same way. We begin with a lemma as before.

LEMMA II.

$$\phi(n, \delta) \equiv \sum_{r=1}^{\delta} \frac{[\delta]_r}{[r]} [rn + r]\,[n + \delta - r]_{\delta - r} - \delta\,[n + \delta]_\delta = 0.$$

This identity is easy to prove directly for the values 1, 2, 3 of δ, and all values of n; it is also obviously true for $n = 0$, and all values of δ. We proceed to establish a double induction.

Now

$$[\delta]_r = [\delta - 1]_r + z^{\delta - r} [\delta - 1]_{r-1} [r];$$

and

$$[rn + r] = [rn + 2r] - z^{rn+r} [r];$$

hence

$$\phi(n,\delta) - [n+1]\,\phi(n+1,\delta-1) = \sum_{r=1}^{\delta}\{-z^{nr+r}[\delta-1]\,[n+\delta-r]_{\delta-r} + z^{\delta-r}[\delta-1]_{r-1}[rn+r]\,[n+\delta-r]_{\delta-r}\} - [n+\delta]_{\delta}.$$

The conventions are adopted here that

$$[\delta-1]_{\delta}=0, \qquad [\delta-1]_{-1}=0.$$

Then

$$z^{\delta-r}[\delta-1]_{r-1}[rn+r] - z^{nr+r}[\delta-1]_r = [\delta-1]_{r-1}[rn+r] - [\delta-1]_r$$

and hence

$$\phi(n,\delta) - [n+1]\,\phi(n+1,\delta-1) = \sum_{r=0}^{\delta}\{-[\delta-1]_r[n+\delta-r]_{\delta-r} + [\delta-1]_{r-1}[rn+r]\,[n+\delta-r]_{\delta-r}\}.$$

Again

$$\begin{aligned}&-[\delta-1]_r[n+\delta-r]_{\delta-r} + [\delta-1]_{r-1}[rn+r]\,[n+\delta-r]_{\delta-r}\\ &+ z^{n+1}[\delta-1]_r[n+\delta-r]_{\delta-r} - z^{n+1}[\delta-1]_{r-1}[rn+r-n-1]\,[n+\delta-r]_{\delta-r}\\ &= [n+1]\{-[\delta-1]_r[n+\delta-r]_{\delta-r} + [\delta-1]_{r-1}[n+\delta-r]_{\delta-r}\}\\ &= [n+1]\,z^{\delta-r}[\delta-1]_{r-1}[n+\delta-r]_{\delta-r}.\end{aligned}$$

Hence

$$\phi(n,\delta) - [n+1]\,\phi(n+1,\delta-1) - z^{n+1}[\delta-1]\{\phi(n,\delta-1) - [n+1]\,\phi(n+1,\delta-2)\} = [n+1]\{\sum_{r=1}^{\delta} z^{\delta-r}[\delta-1]_{r-1}[n+\delta-r]_{\delta-r} - [n+\delta]_{\delta-1}\}.$$

Further

$$z^{\delta-r}[n+\delta-r]_{\delta-r}[\delta-1]_{r-1} = z^{\delta-r}[n-1+\delta-r]_{\delta-r}[\delta-1]_{r-1} + z^{n+1}[\delta-1]\,.\,z^{\delta-r-1}[n+\delta-1-r]_{\delta-1-r}[\delta-2]_{r-1};$$

and

$$[n+\delta]_{\delta-1} = [n-1+\delta]_{\delta-1} + z^{n+1}[\delta-1]\,[n+\delta-1]_{\delta-2}.$$

We deduce at once the identity

$$\begin{aligned}&\phi(n,\delta) - [n+1]\,\phi(n+1,\delta-1) - z^{n+1}[\delta-1]\{\phi(n,\delta-1) - [n+1]\,\phi(n+1,\delta-2)\}\\ &= \frac{[n+1]}{[n]}\big(\phi(n-1,\delta) - [n]\,\phi(n,\delta-1) - z^n[\delta-1]\{\phi(n-1,\delta-1) - [n]\,\phi(n,\delta-2)\}\big)\\ &\quad + z^{n+1}[\delta-1]\big(\phi(n,\delta-1) - [n+1]\,\phi(n+1,\delta-2) - z^{n+1}[\delta-2]\{\phi(n,\delta-2) - [n+1]\,\phi(n+1,\delta-2)\}\big).\end{aligned}$$

This equation enables us to deduce that $\phi(n,\delta)=0$, provided this is true for all lower values of δ and any value of n and also for the given value of δ and values of

n, less than that considered. The induction is thus complete for we already know the truth of the result when $n = 0$ and δ has any value, and also when $\delta = 1, 2$, or 3 and n has any value.

§ 21. THEOREM XIV—*When in the generalized form* Φ_δ *of the symmetric group the symbol* s_r *for a cycle of degree r is replaced by*

$$\frac{[(n+1)\,r]}{[r]},$$

then

$$\Phi_\delta = \frac{\delta\,!\;[n+\delta]_\delta}{[\delta]\,!}.$$

This is easily verified for $\delta = 1, 2, 3$ and we proceed by induction.

Let P_r be that part of the expression which arises from those operations of the group which contain a particular letter b in a cycle of degree r. Then

$$P_r = (\delta - 1)\,!\;\frac{[n+\delta-r]_{\delta-r}\,[(n+1)\,r]}{[r]\;[\delta-r]\,!}.$$

Hence

$$\Phi_\delta = \sum_{r=1}^{\delta} P_r = \delta\,!\;\frac{[n+\delta]_\delta}{[\delta]\,!},$$

on using the lemma.

The arguments of § 19 may now be repeated to establish the following :—

THEOREM XV—

$$\delta\,!\;z^{\varpi_a} f_a^{\,n}(z) = \sum_\beta h_\beta\,\chi_\beta^{\,a}\,[\delta]\,!\;\prod_{r=1}^{k}\frac{[(n+1)\,\beta_r]}{[\beta_r]}$$

and

$$[\delta]\,!\;\prod_{r=1}^{k}\frac{[(n+1)\,\beta_r]}{[\beta_r]} = \sum_a \chi_\beta^{(a)}\,z^{\varpi_a} f_a^{\,n}(z).$$

V—THE COMPOUND SYMMETRIC GROUP

§ 22. The group generated by the permutations

$$(a_r\,a_{r+1})\,(b_r\,b_{r+1}), \qquad r = 1, 2, \ldots \delta - 1$$

will be called the compound symmetric group of degree δ. It possesses the same number of irreducible representations as the ordinary symmetric group and they are of identically the same form ; we may write them

$$T^{(ab)}_{\alpha_1, \alpha_2, \ldots \alpha_h}.$$

Further corresponding to such a representation there are semi-normal units, which will be written

$$\Pi_{\rho,\sigma}.$$

A unit $\Pi_{\rho,\sigma}$ is a linear function of the permutations of the compound symmetric group. Every such permutation is a product of a permutation of the a's by a permutation of the b's. Let $\varpi_{r,s}$, $\varpi'_{u,v}$ be semi-normal units of any representations of the a's and of the b's respectively. Then

$$\Pi_{\rho,\sigma} = \Sigma\, \lambda\varpi_{r,s}\varpi'_{u,v},$$

where λ is numerical.

For our purposes, it is important to calculate this expression for $\Pi_{\rho,\sigma}$ or at least for

$$T^{(ab)} = \sum_{\rho=1}^{f} \Pi_{\rho,\rho}\,.$$

In the first place consider the representation

$$T_{\delta}^{(ab)} = \{A_1A_2 \ldots A_\delta\}\,;$$

that is, the sum of all the permutations of the compound symmetric group divided by their number. In Q.S.A., VIII*, § 20, it was shown that

$$\{A_1A_2 \ldots A_\delta\}\, \varpi_{r,s}\, \varpi'_{u,v} = 0,$$

unless the units $\varpi_{r,s}$, $\varpi'_{u,v}$ are identical, except for the different letters employed. Hence

(IX)
$$\begin{aligned}\{A_1\, A_2 \ldots A_\delta\} &= \Sigma\lambda\varpi_{r,s}\, \varpi'_{u,v}\\ &= \Sigma\lambda\, \{A_1\, A_2 \ldots A_\delta\}\, \varpi_{r,s}\, \varpi'_{u,v}\\ &= \Sigma\lambda_{r,s}\, \varpi_{r,s}\, \varpi'_{r,s}\,.\end{aligned}$$

Now multiply each side by

$$(A_kA_{k+1}) = (a_ka_{k+1})\ (b_kb_{k+1})$$

on the left, and let us suppose that the tableau F_r is changed by (a_ka_{k+1}) into the later tableau F_t. Then (using orthogonal units)

$$\begin{aligned}\Sigma\lambda_{rs}\varpi_{rs}\varpi'_{rs} = \Sigma\lambda_{rs}\,&[-\rho^{-1}\varpi_{rs} + \rho^{-1}\sqrt{(\rho^2-1)}\,\varpi_{ts}]\,[-\rho^{-1}\varpi'_{rs} + \rho^{-1}\sqrt{(\rho^2-1)}\,.\,\varpi'_{ts}]\\ &+ \Sigma\lambda_{ts}\,[\rho^{-1}\sqrt{(\rho^2-1)}\,\varpi_{rs} + \rho^{-1}\varpi_{ts}]\,[\rho^{-1}\sqrt{(\rho^2-1)}\,\varpi'_{rs} + \rho^{-1}\varpi'_{ts}]\,;\end{aligned}$$

hence $\lambda_{rs} = \lambda_{ts}$. Similarly, by right-hand multiplication we find $\lambda_{sr} = \lambda_{st}$. Thus all the coefficients are the same, for any particular representation $T_{\alpha_1, \alpha_2 \ldots \alpha_h}$ to which $\varpi_{r,s}$ and $\varpi'_{r,s}$ respectively belong.

Consider a particular representation T then since

$$\varpi_{r,s}\varpi'_{r,s}\,.\,\varpi_{u,v}\varpi'_{u,v} = 0,$$

unless $s = u$, and

$$\varpi_{r,s}\varpi'_{r,s}\,.\,\varpi_{s,v}\varpi'_{s,v} = \varpi_{r,v}\varpi'_{r,v}\,,$$

the sum $\Sigma\lambda_{rs}\varpi_{r,s}\varpi'_{r,s}$ can be expressed as the matrix $[\lambda_{rs}]$ of order f for this representation. In our case all the elements λ_{rs} have the same value.

* YOUNG, 'Proc. London Math. Soc.,' vol. 37, p. 441 (1934).

Now take the square of each side of equation (IX), we obtain

$$\{A_1A_2 \ldots A_\delta\} = \Sigma Q^2 = \Sigma Q,$$

where Q is represented by the matrix $[\lambda_{rs}]$; and hence

$$Q^2 = f\lambda_{rs} Q.$$

Hence

$$f\lambda_{rs} = 1,$$

and

(X) $$\{A_1A_2 \ldots A_\delta\} = \Sigma f^{-1} \varpi_{rs} \varpi'_{rs}.$$

Here the Σ extends to every unit of every representation.

§ 23. There are certain fairly obvious restrictions to the coefficients in the equation

$$\Pi_{\rho,\sigma} = \Sigma \lambda \varpi_{r,s} \varpi'_{u,v}.$$

We take orthogonal units.

(i) $\Pi_{\rho,\sigma}$ is a linear function of the form

$$\Sigma \mu \tau \tau',$$

where μ is numerical, τ is a permutation of the a's, and τ' the same permutation of the b's. We deduce at once

$$\Pi_{\rho,\sigma} = \Sigma \lambda \, (\varpi_{r,s} \varpi'_{u,v} + \varpi_{u,v} \varpi'_{r,s}).$$

(ii) $$\Pi_{\sigma,\rho} = \Sigma \lambda \, (\varpi_{s,r} \varpi'_{v,u} + \varpi_{v,u} \varpi'_{s,r}).$$

(iii) $$\Pi_{\rho,\rho} = \Sigma \lambda \, (\varpi_{r,s} \varpi'_{u,v} + \varpi_{u,v} \varpi'_{r,s} + \varpi_{s,r} \varpi'_{v,u} + \varpi_{v,u} \varpi'_{s,r}).$$

(iv) When every transposition both of the a's and b's is changed in sign, the permutations of the compound group are all unchanged; hence, if the suffix letters of the units be also taken to define the corresponding tableaux, and $F_{r'}$ be the tableau obtained by interchanging rows and columns in F_r,

$$\Pi_{\rho,\sigma} = \Sigma \lambda \, (\varpi_{r,s} \varpi'_{u,v} + \varpi_{u,v} \varpi'_{r,s} + \varpi_{r',s'} \varpi'_{u',v'} + \varpi_{u',v'} \varpi'_{r',s'}).$$

(v) When every transposition of the a's is changed in sign, but the transpositions of the b's are unchanged, we obtain

$$\Pi_{\rho'\sigma'} = \Sigma \lambda \, (\varpi_{r',s'} \varpi'_{u,v} + \varpi_{u',v'} \varpi'_{r,s} + \varpi_{r,s} \varpi'_{u',v'} + \varpi_{u,v} \varpi_{r',s'}).$$

The last consideration enables us to write down without further enquiry the value

$$T_{1\delta}^{(ab)} = \Sigma f^{-1} \varpi_{rs} \varpi'_{r',s'}.$$

It is to be noticed that the orders f, f' of conjugate representations are the same.

§ 24. Let $\Gamma^{(a)}$ represent as before the operation of taking the sum of the δ ! terms obtained by permuting the letters *a* in the operand. Then

(XI) $$\Gamma^{(a)}T^{(ab)}_{\alpha_1,\alpha_2\ldots\alpha_h} = \sum_\beta \mu_\beta t_\beta^{(a)} t_\beta^{(b)}$$
$$= \sum_{\epsilon,\zeta} \nu_{\epsilon,\zeta} T_\epsilon^{(a)} T_\zeta^{(b)},$$

where μ_β, $\nu_{\epsilon,\zeta}$ are numerical, t_β is the sum of the members of a particular conjugate set of permutations of the letters concerned, ε, ζ define certain definite irreducible representations, and $\nu_{\epsilon,\zeta} = \nu_{\zeta,\epsilon}$. Hence, when

$$T_\alpha^{(ab)} = \Sigma \lambda_{rs,uv} \varpi_{r,s} \varpi'_{u,v},$$

we find

$$\Gamma^{(a)} \Sigma \lambda_{rs,uv} \varpi_{r,s} \varpi'_{u,v} = \sum_{\epsilon,\zeta} \nu_{\epsilon,\zeta} T_\epsilon^{(a)} T_\zeta^{(b)}.$$

The equation (XI) will be written

(XII) $$\Gamma^{(a)} T_\alpha^{(ab)} = \Sigma f_{\alpha\beta\gamma} T_\beta^{(a)} T_\gamma^{(b)}.$$

§ 25. Frobenius* discussed a closely allied problem, and his results come in very usefully here. Consider two linear substitution groups, which are representations of the same abstract group. Let

$$u_\alpha = \sum_\beta a_{\alpha\beta} v_\beta \qquad (\alpha, \beta = 1, 2, \ldots f)$$

and

$$u'_\gamma = \sum_\delta a'_{\gamma\delta} v'_\delta \qquad (\gamma, \delta = 1, 2, \ldots f')$$

be corresponding substitutions in the two groups. A third group is constructed by compounding the substitutions of the first two, in this the substitution corresponding to those written above is

$$u_\alpha u'_\gamma = \sum_{\beta,\delta} a_{\alpha\beta} a'_{\gamma,\delta} v_\beta v'_\delta.$$

This group is a representation of the same abstract group as a linear substitution group with ff' variables. In general, this third group is reducible ; let Φ be its group determinant, then the reduction may be expressed by the equation

$$\Phi = \prod_\mu \Phi_{\mu'}{}^{f_{\kappa\lambda\mu}},$$

where the suffixes κ, λ, μ define irreducible representations of the group, μ and μ' being conjugate imaginary representations (they are the same when the representation is real), and κ, λ define the representations by the two groups from which we started.

* 'S. B. berl. math. Ges.' (volumes are not numbered), p. 330 (1899).

FROBENIUS proved that $f_{\kappa\lambda\mu}$ is a positive integer or zero unchanged in value by any permutation of the suffixes. Its value is given by

$$hf_{\kappa\lambda\mu} = \sum_{R} \chi^{(\kappa)}(R)\, \chi^{(\lambda)}(R)\, \chi^{(\mu)}(R)\ ;$$

where h is the order of the group, and $\chi^{(\kappa)}(R)$ is the characteristic of the group element R in the representation κ.

Also

$$f_{\kappa} f_{\lambda} = \sum_{\mu} f_{\kappa\lambda\mu} f_{\mu},$$

$$\Sigma f_{\kappa\lambda\mu}{}^2 = \sum_{\rho} h/h_{\rho},$$

where h_{ρ} is the number of operations in the ρ^{th} conjugate class.

§ 26. The problem of FROBENIUS is not quite identical with that considered here, and so it is not permissible merely to quote his result without further enquiry. Let $\chi^{\alpha}(R)$ be the characteristic of the permutation R in the representation T_{α} of the symmetric group of degree δ. When the permuted letters have to be expressed, we write $R^{(a)}$ or $R^{(b)}$ as the case may be, and for the compound group

$$R^{(ab)} = R^{(a)} R^{(b)}.$$

Then from equation (XII)

$$\frac{\delta\,!}{h_{R}} \chi^{(\alpha)}(R) = \sum_{\beta,\,\gamma} f_{\alpha\beta\gamma}\, \chi^{(\beta)}(R)\, \chi^{(\gamma)}(R),$$

and

$$0 = \sum_{\beta,\,\gamma} f_{\alpha\beta\gamma}\, \chi^{(\beta)}(R)\, \chi^{(\gamma)}(S),$$

when R and S are not conjugate. Moreover, these equations are the necessary and sufficient conditions for the truth of (XII). FROBENIUS found the equation

(XIII) $$\chi^{\kappa}(R)\, \chi^{\lambda}(R) = \sum_{\mu} f_{\kappa\lambda\mu'}\, \chi^{(\mu)}(R'),$$

and from which he derived the value of $f_{\kappa\lambda\mu}$. Multiply (XIII) by $\chi^{(\lambda)}(S)$, and sum the equation for all values of λ, we have when S is not conjugate to R

$$0 = \sum_{\lambda\mu} f_{\kappa\lambda\mu'}\, \chi^{(\lambda)}(S)\, \chi^{(\mu)}(R)\ ;$$

and when S = R

$$\frac{h}{h_{R}} \chi^{(\kappa)}(R) = \sum_{\lambda\mu} f_{\kappa\lambda\mu'}\, \chi^{\lambda}(R)\, \chi^{\mu}(R).$$

These equations are identical with (XII), when it is remembered that the representation μ is the same as the conjugate imaginary representation for the symmetric group. Hence the quantity $f_{\alpha\beta\gamma}$ of the present discussion is the same as that introduced by FROBENIUS. We therefore may use his results.

VI—The Function $f_\delta(x, y)$

§ 27. We define this function in terms of the functions of Section I, thus:

$$f_\delta(x, y) = \Sigma\, x^{\varpi} y^{\varpi} f_{\alpha_1, \alpha_2, \dots \alpha_h}(x)\, f_{\alpha_1, \alpha_2, \dots \alpha_h}(y),$$

the summation extends to all the representations of the symmetric group of δ letters.

The importance here of this function lies in the fact that the generating function for ternary perpetuants of degree δ for a single form may be written

$$\frac{(1-x)(1-y)(x-y) f_\delta(x, y)}{x \,.\, \{[\delta]\,!\}_x \{[\delta]\,!\}_y},$$

where the suffix after the bracket indicates the particular variable in the function.

The attempt is made to express this in the form

$$\frac{\phi(x, xy)}{\psi(x, xy)} - \frac{y\phi(y, xy)}{x\psi(y, xy)},$$

where ϕ, ψ are rational integral functions; thus the function would be separated into two parts of which obviously the first only concerns the perpetuant problem in hand. The attempt is successful up to degree 5; and indications are obtained as to what may be expected in general.

Theorem XVI—$\delta\,! f_\delta(x, y) = \sum_\beta h_\beta X_\beta Y_\beta$, *where* β *stands for a partition* $\beta_1, \beta_2, \dots \beta_k$ *of* δ; $X_\beta = \left\{\dfrac{[\delta]\,!}{[\beta_1][\beta_2]\dots[\beta_k]}\right\}_x$, Y_β *is the same function of* y, *and* h_β *is the number of members of the symmetric group of degree* δ *which belong to the conjugate class defined by* β.

Consider the effect of replacing each "a" cycle of degree r by $(1-x^r)^{-1}$ and each "b" cycle of degree r by $(1-y^r)^{-1}$. The compound symmetric group of degree δ, $\delta\,!\,T_\delta$, becomes

$$\Sigma\, \frac{h_\beta X_\beta Y_\beta}{\{[\delta]\,!\}_x \{[\delta]\,!\}_y}.$$

Also as in §§ 15, 19, we see that $\varpi_{r,s}$ becomes zero unless $r = s$, and then it has the value $x^{\varpi_\alpha} f_\alpha(x)/\{[\delta]\,!\}_x$, where T_α is the representation to which the unit $\omega_{r,r}$ belongs. Thus, from equation (X), § 22, T_δ becomes, after making this replacement, $f_\delta(x, y)/(\{[\delta]\,!\}_x \{[\delta]\,!\}_y)$. The theorem is proved by equating the two values obtained for T_δ.

§ 28. A whole set of functions may now be defined analogous to the functions $f_{\alpha_1, \alpha_2, \dots \alpha_h}(x)$, as follows:—

$$f_{\alpha_1, \alpha_2, \dots \alpha_h}(x, y) = \sum_{\beta, \gamma} f_{\alpha\beta\gamma}\, x^{\varpi_\beta} y^{\varpi_\gamma} f_\beta(x)\, f_\gamma(y).$$

That $f_\delta(x, y)$ is a particular case will be seen at once.

THEOREM XVII—

$$\delta\,!\,f_\alpha(x, y) = \sum_\beta h_\beta \chi_\beta^{(\alpha)} X_\beta Y_\beta.$$

And

$$X_\beta Y_\beta = \sum_\alpha \chi_\beta^{(\alpha)} f_\alpha(x, y).$$

The proof is practically a repetition of §§ 19 and 27.

§ 29. The same processes may be extended. Thus one or both of the sets of letters *a*, *b* may be supposed to represent compound groups—or multiply compound groups. Thus we may have *m* sets of letters

$$a_{r,s}, \quad r = 1, 2, \ldots m, \quad s = 1, 2, \ldots \delta\,;$$

and consider the symmetric group of degree δ formed by permuting all the sets simultaneously in exactly the same way. We obtain functions

$$f_{\alpha_1, \alpha_2, \ldots \alpha_h}(x_1, x_2, \ldots x_m).$$

These may be obtained in the first place by grouping the *m* sets into two groups and using the above arguments; and then we can break these groups up further. It is to be noticed that the same final result must be obtained however the *m* sets are thus divided. Or else following FROBENIUS we may use his numerical function

$$f_{\alpha\beta\gamma\ldots},$$

with $m+1$ suffixes, and write

$$f_\alpha(x_1, x_2, \ldots x_m) = \sum_{\beta, \gamma} f_{\alpha\beta\gamma\ldots}\, x_1^{\varpi_\beta} x_2^{\varpi_\gamma} \ldots f_\beta(x_1) f_\gamma(x_2) \ldots$$

FROBENIUS* proved that

$$f_{\alpha\beta\gamma\epsilon\ldots} = \sum_R \chi^{(\alpha)}(R)\, \chi^{(\beta)}(R)\, \chi^{(\gamma)}(R)\, \chi^{(\epsilon)}(R) \ldots.$$

Then, as before, we obtain

THEOREM XVIII—

$$\delta\,!\,f_\alpha(x_1, x_2, \ldots x_m) = \sum_\beta h_\beta \chi_\beta^{(\alpha)} X_\beta^{(1)} X_\beta^{(2)} \ldots X_\beta^{(m)}.$$

And

$$X_\beta^{(1)} X_\beta^{(2)} \ldots X_\beta^{(m)} = \sum_\alpha \chi_\beta^{(\alpha)} f_\alpha(x_1, x_2, \ldots x_m).$$

THEOREM XIX—*When* T_α, T_{α}', *are conjugate representations*

$$f_{\alpha'}(x, y) = y^{\binom{\delta}{2}} f_\alpha(x, y^{-1}) = x^{\binom{\delta}{2}} f_\alpha(x^{-1}, y).$$

It was shown § 6 that

$$z^{\varpi'} f_{\alpha'}(z) = z^{\binom{\delta}{2}} [z^{-\varpi} f_\alpha(z^{-1})].$$

Also it follows from (v), § 23, that

$$f_{\alpha\beta\gamma} = f_{\alpha'\beta\gamma'} = f_{\alpha'\beta'\gamma} = f_{\alpha\beta'\gamma'}.$$

* 'S. B. berl. math. Ges.,' p. 330 (1899).

The result follows at once on replacing y by y^{-1} in the equation

$$f_\alpha(x, y) = \sum_{\beta\gamma} f_{\alpha\beta\gamma} x^{\varpi_\beta} y^{\varpi_\gamma} f_\beta(x) f_\gamma(y).$$

§ 30. The next problem is to separate the function

$$\frac{(1-x)(1-y)(x-y) f_\delta(x, y)}{x\{[\delta]\,!\}_x \{[\delta]\,!\}_y} = \frac{\phi(x, xy)}{\psi(x, xy)} - \frac{y\phi(y, xy)}{x\psi(y, xy)}.$$

For $\delta = 2, 3, 4$ the problem is solved quite easily. We use the negative symmetric group $\{xy\}'$ thus :—

$$(x-y) f_2(x, y) = \{xy\}'(x + x^2 y) = \{xy\}' x(1-y^2);$$

hence

$$\frac{(x-y) f_2(x, y)}{x(1-x^2)(1-y^2)} = \frac{1}{1-x^2} - \frac{y}{x(1-y^2)}.$$

For $\delta = 3$,

$$(x-y) f_3(x, y)(1-x^2y^2) = \{xy\}'(1-y^2)(1-y^3)(x + yx^5),$$

hence

$$\frac{(x-y) f_3(x, y)}{x(1-x^2)(1-x^3)(1-y^2)(1-y^3)} = \frac{1 + yx^4}{(1-x^2)(1-x^3)(1-x^2y^2)} - \frac{y + xy^5}{x(1-y^2)(1-y^3)(1-x^2y^2)}.$$

For $\delta = 4$,

$$\begin{aligned}(x-y) f_4(x, y)(1-x^2y^2)(1-x^3y^3)(1-x^4y^4) \\ = \{xy\}'(1-y^2)(1-y^3)(1-y^4)[x + y(x^5 + x^6 + x^7 - x^9) + y^2(x^4 + x^5) \\ - y^3(x^4 - x^5 - x^6 - x^7 + x^9) + y^4(x^7 - x^9 - x^{10} - x^{11} + x^{12}) \\ - y^5(x^{11} + x^{12}) + y^6(x^7 - x^9 - x^{10} - x^{11}) - y^7 x^{15}].\end{aligned}$$

There is no need to put down the second part of the function

$$x^{-1}(1-x)(1-y)(x-y) f_4(x, y)/\{[4]\,!\}_x \{[4]\,!\}_y,$$

the first part is

$$\begin{aligned}\{(1-x^2)(1-x^3)(1-x^4)(1-x^2y^2)(1-x^3y^3)(1-x^4y^4)\}^{-1} \\ \times [1 + yx^4(1 + x + x^2 - x^4) + y^2x^3(1+x) - y^3x^3(1 - x - x^2 - x^3 + x^5) \\ + y^4x^6(1 - x^2 - x^3 - x^4 + x^5) - y^5x^{10}(1+x) \\ + y^6x^6(1 - x^2 - x^3 - x^4) - y^7x^{14}].\end{aligned}$$

§ 31. The difficulty of calculating the function increases rapidly with δ. It appears to be easiest to use the form of $f_\delta(x, y)$ obtained in § 27 when $\delta > 4$; for instead of a single function $f_\delta(x, y)$ to be discussed, there are several much simpler functions which can be dealt with separately. There is the additional advantage that the solution of the problem for $f_\delta(x, y)$ obtained in this form can be at once applied to $f_\alpha(x, y)$.

Theorem XVI gives us :—

$$f_3(x, y) = \tfrac{1}{6}(1 + x)(1 + x + x^2)(1 + y)(1 + y + y^2) + \tfrac{1}{3}(1 - x)(1 - x^2)(1 - y)(1 - y^2) + \tfrac{1}{2}(1 - x^3)(1 - y^3)$$

$$f_4(x, y) = \tfrac{1}{24}X_{1^4}Y_{1^4} + \tfrac{1}{4}X_4Y_4 + \tfrac{1}{8}X_{2^2}Y_{2^2} + \tfrac{1}{3}X_{3,1}Y_{3,1} + \tfrac{1}{4}X_{2,1^2}Y_{2,1^2}$$

$$f_5(x, y) = \tfrac{1}{120}X_{1^5}Y_{1^5} + \tfrac{1}{5}X_5Y_5 + \tfrac{1}{4}X_{4,1}Y_{4,1} + \tfrac{1}{6}X_{3,2}Y_{3,2} + \tfrac{1}{6}X_{3,1^2}Y_{3,1^2} + \tfrac{1}{8}X_{2^2,1}Y_{2^2,1} + \tfrac{1}{12}X_{2,1^3}Y_{2,1^3}.$$

In the first place, we have to consider

$$\frac{x - y}{(1 - x)^{\delta-1}(1 - y)^{\delta-1}}.$$

Let us write z for xy.

Then

$$(1 - z) = x(1 - y) + y(1 - x) + (1 - x)(1 - y),$$

and

$$x - y = x(1 - y) - y(1 - x).$$

Let us write

$$(x - y)(1 - z)^{2\delta-5} = (1 - y)^{\delta-1} xP_\delta(x, z) - (1 - x)^{\delta-1} yP_\delta(y, z);$$

for the index $2\delta - 5$ of z is both necessary and sufficient in order that an integral function $P_\delta(x, z)$ may be found which satisfies the condition. Then also

$$(x - y)(1 - z)^{2\delta-5} = (1 - z)^2 [(1 - y)^{\delta-2} xP_{\delta-1}(x, z) - (1 - x)^{\delta-2} yP_{\delta-1}(y, z)],$$

hence

$$(1 - y)^{\delta-2}[(1 - z)^2 xP_{\delta-1}(x, z) - (x - z)P_\delta(x, z)]$$
$$= (1 - x)^{\delta-2}[(1 - z)^2 yP_{\delta-1}(y, z) - (y - z)P_\delta(y, z)]$$
$$= (1 - x)^{\delta-2}(1 - y)^{\delta-2}\Phi_\delta(z);$$

for it is a function of degree $\delta - 2$ in x and in y and has a factor $(1 - x)^{\delta-2}$, and also $(1 - y)^{\delta-2}$.

Hence

$$(1 - z)^2 xP_{\delta-1}(x, z) - (x - z)P_\delta(x, z) = (1 - x)^{\delta-2}\Phi_\delta(z).$$

Now put $x = z$,

$$\Phi_\delta(z) = z(1 - z)^{-\delta+4} P_{\delta-1}(z, z).$$

Thus

$$(x - z)P_\delta(x, z) = (1 - z)^2 xP_{\delta-1}(x, z) - z(1 - x)^{\delta-2}(1 - z)^{-\delta+4} P_{\delta-1}(z, z),$$

gives a scale of relation to calculate P_δ.

The following results are obtained :—

$$P_3(x, z) = 1,\ P_4(x, z) = 1 - xz,$$
$$P_5(x, z) = 1 + z - 4xz + zx^2 + z^2x^2.$$

§ 32. The second term in $f_5(x, y)$, due to X_5Y_5, is

$$\frac{(1-x)(1-y)(x-y)}{x(1-x^5)(1-y^5)}.$$

Then

$$\{xy\}' x(1-x)(1-y)(1-z^5) = \{xy\}'(1-y^5)x[1-x+z(1-x^3) + z^2x^2(1-x)].$$

And

$$\frac{(1-x)(1-y)(x-y)}{x(1-x^5)(1-y^5)} = \frac{1-x+z(1-x^3)+z^2x^2(1-x)}{(1-x^5)(1-z^5)} - y\frac{1-y+z(1-y^3)+z^2y^2(1-y)}{x(1-y^5)(1-z^5)}.$$

The third term $X_{4,1}$, $Y_{4,1}$, belongs to a very simple class, viz. :—

$$\frac{x-y}{x(1-x^n)(1-y^n)}.$$

Here we may use

$$1-z^n = x^n(1-y^n) + y^n(1-x^n) + (1-x^n)(1-y^n).$$

In our case $n = 4$ and the result is

$$\frac{1-zx^2}{(1-x^4)(1-z^4)} - y\frac{1-zy^2}{x(1-y^4)(1-z^4)}.$$

The fourth and fifth terms may be taken together, for

$$(1-x)(1-y)(x-y)(X_{3,2}Y_{3,2} + X_{3,1^2}Y_{3,1^2})/(x\{[5]!\}_x\{[5]!\}_y) = 2\frac{(x-y)(1+z)}{x(1-x^2)(1-x^3)(1-y^2)(1-y^3)}.$$

It is necessary to introduce the factors $(1-z^3)(1-z^6)$ and the x, z function is

$$2\{(1-x^2)(1-x^3)(1-z^3)(1-z^6)\}^{-1} \times [1 - zx(1-x^2) + z^3x(1-x) - z^5(1-x^2) - z^6x^3].$$

The sixth term $X_{2^2,1}Y_{2^2,1}$ is

$$\frac{x-y}{x(1-x^2)^2(1-y^2)^2},$$

and the corresponding x, z function is

$$\frac{1-z}{(1-x^2)^2(1-z^2)^2} - \frac{z(1-z)^2}{(1-x^2)(1-z^2)^3}.$$

Lastly, we have $X_{2, 1^3} Y_{2, 1^3}$, and

$$\frac{x - y}{x(1-x)^2(1-x^2)(1-y)^2(1-y^2)},$$

with the x, z function

$$\frac{x^2}{(1-x)^2(1-x^2)(1-z)(1-z^2)} + \frac{2x(1-x)+z(1-x^2)}{(1-x)^2(1-x^2)(1-z)(1-z^2)^2} + \frac{1-z}{(1-x^2)(1-z^2)^3}.$$

§ 33. Before giving the final result for the case $\delta = 5$. There is a general remark that can now be made clear about these functions. The form of result that is sought is as in § 27 :—

$$\frac{(1-x)(1-y)(x-y)f_\delta(x,y)}{x\{[\delta]!\}_x\{[\delta]!\}_y} = \frac{(1-x)\phi(x,z)}{\{[\delta]!\}_x\psi(z)} - \frac{(1-y)y\phi(y,z)}{x\{[\delta]!\}_y\psi(z)}.$$

Write x^{-1} for x, and y^{-1} for y, then z becomes z^{-1}; also

$$x^{\binom{\delta}{2}}y^{\binom{\delta}{2}}f_\delta(x^{-1}, y^{-1}) = f_\delta(x,y),$$

by Theorem III, § 6. Then

$$(-)^{\delta-1}\left\{\frac{x^{\binom{\delta+1}{2}-1}(1-x)\phi(x^{-1},z^{-1})}{\{[\delta]!\}_x\psi(z^{-1})} - \frac{y^{\binom{\delta+1}{2}-2}x(1-y)\phi(y^{-1},z^{-1})}{\{[\delta]!\}_y\psi(z^{-1})}\right\}$$

$$= -\frac{x^{\delta-1}y^{\delta-2}(1-x)(1-y)(x-y)f_\delta(x,y)}{\{[\delta]!\}_x\{[\delta]!\}_y}$$

$$= -x^\delta y^{\delta-2}\left\{\frac{(1-x)\phi(x,z)}{\{[\delta]!\}_x\psi(z)} - \frac{y(1-y)\phi(y,z)}{x\{[\delta]!\}_y\psi(z)}\right\},$$

whence

$$x^{\binom{\delta+1}{2}-3}\phi(x^{-1},z^{-1})\psi(z) = (-)^\delta z^{\delta-2}\phi(x,z)\psi(z^{-1}).$$

Now ϕ always contains a term independent of x and thus the degree of ϕ in x is exactly $\binom{\delta+1}{2} - 3$. Owing to the equation

$$1 - z^n = y^n(1-x^n) + x^n(1-y^n) + (1-x^n)(1-y^n),$$

it is clear that it is always possible to choose ψ as a product of binomial factors $1 - z^n$, just as has been done for the cases discussed. Let p be the number of binomial factors in ψ and q its total degree, then

$$\psi(z) = (-)^p z^q \psi(z^{-1}).$$

The degree of ϕ in z is $q - \delta + 2$, and, in fact,

$$\text{(XIV)} \qquad \phi(x,z) = (-)^{\delta+p}x^{\binom{\delta+1}{2}-3}z^{q-\delta+2}\phi(x^{-1},z^{-1}).$$

§ 34. For $\delta = 5$ we insert the numerical coefficients in the results of §§ 31, 32 to obtain the x, z part of

$$\frac{(1-x)(1-y)(x-y) f_5(x,y)}{x\{[5]!\}_x \quad \{[5]!\}_y},$$

viz. :—

$$\frac{1+z(1-4x+x^2)+z^2x^2}{120(1-x)^4(1-z)^5}+\frac{1-x+z(1-x^3)+z^2x^2(1-x)}{5(1-x^5)(1-z^5)}$$

$$+\frac{1-zx^2}{4(1-x^4)(1-z^4)}+\frac{1-zx(1-x^2)+z^3x(1-x)-z^5(1-x^2)-z^6x^3}{3(1-x^2)(1-x^3)(1-z^3)(1-z^6)}$$

$$+\frac{1-z}{8(1-x^2)^2(1-z^2)^2}-\frac{z(1-z)^2}{8(1-x^2)(1-z^2)^3}$$

$$+\frac{x^2}{12(1-x)^2(1-x^2)(1-z)(1-z^2)}$$

$$+\frac{2x(1-x)+z(1-x^2)}{12(1-x)(1-x^2)(1-z)(1-z^2)^2}+\frac{1-z}{12(1-x^2)(1-z^2)^3}.$$

In the final result the denominator is

$$(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6),$$

and extraneous numerical factors must disappear—a useful check on the arithmetic. The numerator we call $\phi(x, z)$ and for this equation (XIV) gives

$$\phi(x, z) = x^{12}z^{17}\phi(x^{-1}, z^{-1}).$$

In consequence of this equation it is only necessary to give the terms up to z^8 the rest can be written down at once ; then

$$\begin{aligned}\phi(x,z) = 1 &+ z(x^3+x^4+2x^5+x^6-x^8-x^9-x^{10}+x^{12}) \\ &+z^2(x+2x^2+x^3+x^4) \\ &+z^3(-1+2x+2x^2+3x^3+2x^4-x^6-x^7-x^8+x^{10}) \\ &+z^4(1+2x+3x^2+2x^3+x^4-2x^5-2x^6-2x^7+x^9+x^{10}+x^{11}-x^{12}) \\ &+z^5(-1+2x+3x^2+4x^3+4x^4+x^5-4x^6-5x^7-3x^8-2x^9+x^{10}+2x^{11}+x^{12}) \\ &+z^6(3+3x+3x^2-2x^4-5x^5-5x^6-3x^7-x^8+x^9+2x^{10}+2x^{11}-x^{12}) \\ &+z^7(1+2x+3x^2+3x^3-x^4-3x^5-6x^6-7x^7-5x^8-x^9+x^{10}+2x^{11}+3x^{12}) \\ &+z^8(4+2x+2x^2-x^3-4x^4-6x^5-7x^6-5x^7-2x^8+x^9+x^{10}+3x^{11}+x^{12}) \\ &+\dots+z^{17}x^{12}.\end{aligned}$$

VII—The Generating Function for Ternary Perpetuants of a Single Form

§ 35. It has been proved* that if G_δ be a generating function for ternary forms of any particular type, and Γ_δ be the corresponding generating function for gradients (*i.e.*, coefficient products),

$$G_\delta = (1 - x)(1 - y)(1 - y/x)\,\Gamma_\delta,$$

where the indices of x and y respectively refer as usual to the second and third weights of the form (or of the coefficients).

We therefore proceed to consider the generating function for gradients of degree δ of a single ternary form of infinite order. A gradient is a product

$$A_{p_1 q_1} \ldots A_{p_s q_s},$$

each factor being a coefficient of the form, and the two suffices of the coefficient are its second and third weights.

For gradient types we name the coefficients further, say, with a prefix, to define which of the δ quantics has supplied the particular coefficients. For a single quantic, this fact is substitutionally expressed by the operator $\{{}_1A\ {}_2A \ldots {}_\delta A\}$, the suffices not being here expressed as the prefixes alone are permuted. For binary forms, where there is only one suffix, the fact is equally well and more conveniently expressed by a permutation of the suffices. In ternary forms where there are two suffixes the permutation $({}_rA\ {}_sA)$ is expressed by the compound permutation $(p_r\, p_s)\,(q_r\, q_s)$ of the suffixes. Thus in the ternary case we have to do with the compound symmetric group. We have seen § 22 equation (X) that the compound symmetric group

$$T_\delta^{(pq)} = \Sigma \frac{1}{f} \varpi_{rs}\, \varpi'_{rs}.$$

Let

$$\omega_{rs}\, P$$

be any gradient of the binary coefficients ${}_rA_{pq}$, where ϖ_{rs} is a unit of $T^{(p)}_{\alpha_1 \alpha_2 \ldots \alpha_h}$, then by operation with ϖ_{tr} we may obtain a gradient

$$\omega_{ts}\, P,$$

and thus f distinct gradients of this type. The number of distinct gradients for this representation $T_{\alpha_1 \alpha_2 \ldots \alpha_h}$ has been found to be given by the generating function

$$\{[\delta]\,!\}^{-1} f_{\alpha_1 \ldots \alpha_h} f_{\alpha_1 \ldots \alpha_h}(x) \,.\, x^{\varpi}.$$

Hence the generating function for ternary gradients of a single form of infinite order is

$$\Sigma\ (\{[\delta]\,!\}^{-1})_x\, (\{[\delta]\,!\}^{-1})_y\, f_{\alpha_1 \ldots \alpha_h}(x)\, f_{\alpha_1 \ldots \alpha_h}(y)\ x^{\varpi} y^{\varpi}, = \{[\delta]\,!\}_x^{-1}\, \{[\delta]\,!\}_y^{-1}\, f_\delta(x, y).$$

* Young, 'Proc. Lond. Math. Soc.,' vol. 35, p. 431 (1933).

THEOREM XX—*The generating function for ternary perpetuants of degree* δ *for a single form is*

$$\frac{(1-x)(1-y)(x-y)f_\delta(x,y)}{x\{[\delta]\,!\}_x\,\{[\delta]\,!\}_y}.$$

§ 36. Let us now consider the results of the last section in respect to the generating function. The expansion is an infinite series in both x and y. The terms $x^r y^s$ where $s > r$ do not really concern us, as the seminvariants, with which only we are concerned, never have the third weight greater than the second. In fact, the generating function proper is the part called above

$$G_\delta = \frac{(1-x)\,\phi\,(x,z)}{([\delta]\,!)_x\,\psi\,(z)} = \Sigma\, B_{r,\rho}\, x^r z^\rho.$$

Consider the expression of the perpetuant as a symbolical product, it essentially contains two kinds of factors (abu), (abc); then the index of x in the generating function is the number of factors (abu), and the index of z is the number of factors (abc).

When $\delta = 2$,

$$G_2 = \frac{1}{1-x^2},$$

giving the obviously correct result.

When $\delta = 3$

$$G_3 = \frac{1+zx^3}{(1-x^2)(1-x^3)(1-z^2)}.$$

Here as in the general case the x factors in the denominator refer to (abu) terms which may be taken exactly as in the binary case, viz. :—

$$(a_1a_2u)^{\lambda_1}\,(a_2a_3u)^{\lambda_2}\ldots(a_{\delta-2}a_{\delta-1}u)^{\lambda_{\delta-2}}\,(a_{\delta-1}a_\delta u)^{2\lambda_{\delta-1}},$$

where

$$\lambda_{\delta-1} \geqq \lambda_{\delta-2}\ldots \geqq \lambda_1,$$

see § 8.

The denominator factor $1-z^2$ here refers to a factor $(a_1\,a_2\,a_3)^{2\mu}$; and the numerator term zx^3 corresponds to the form

(XV) $$(a_1\,a_2\,a_3)\,(a_1\,a_2\,u)\,(a_2\,a_3\,u)\,(a_3\,a_1\,u);$$

there are thus two sets of perpetuants of degree 3, one

$$(a_1\,a_2\,a_3)^{2\mu}\,(a_1\,a_2\,u)^{\lambda}\,(a_2\,a_3\,u)^{2\lambda+2\rho};$$

and the other this set multiplied by the form (XV) given by zx^3.

When $\delta = 4$:

$$\begin{aligned}G_4 = \{&1 + z\,(x^3+x^4+x^5-x^7) + z^2\,(x+x^2) - z^3\,(1-x-x^2-x^3+x^5)\\ &+ z^4\,(x^2-x^4-x^5-x^6+x^7) - z^5\,(x^5+x^6) + z^6\,(1-x^2-x^3-x^4) - z^7x^7\}\\ &\times\{(1-x^2)(1-x^3)(1-x^4)(1-z^2)(1-z^3)(1-z^4)\}^{-1}.\end{aligned}$$

The terms independent of x are

$$\frac{1+z^9}{(1-z^2)(1-z^4)(1-z^6)}.$$

It is easy to see that we may take the perpetuants corresponding to the denominator factors to be

$$(a_1\, a_2\, a_3)^2,\ (a_1\, a_2\, a_3)^2\, (a_1\, a_2\, a_4)^2,$$
$$(a_1\, a_2\, a_3)^3\, (a_1\, a_2\, a_4)^3.$$

And that corresponding to the numerator term z^9 to be

$$(a_1\, a_2\, a_3)^6\, (a_1\, a_2\, a_4)^2\, (a_1\, a_3\, a_4).$$

The generating function assures us that all perpetuants of degree four, such that the second and third weights are equal, may be expressed as a sum of terms

$$(a_1 a_2 a_3)^{\lambda_1}\, (a_1 a_2 a_4)^{\lambda_2}\, (a_1 a_3 a_4)^{\lambda_3};$$

where

$$\lambda_1 = 2\mu_1 + 2\mu_2 + 3\mu_3 + 6\varepsilon, \quad \lambda_2 = 2\mu_2 + 3\mu_3 + 2\varepsilon, \quad \lambda_3 = \varepsilon, \quad \varepsilon = 0 \text{ or } 1.$$

It is useless to say anything about the other terms without a careful investigation of them, which has not yet been undertaken.

VIII—Generating Functions for Ternary Forms of Finite Order

§ 37. We concern ourselves with the generating function for gradients.

Theorem XXI—*The generating function* $\Phi_\delta^{(n)}$ *for gradients of degree* δ *of the ternary n-ic is given by*

$$\delta\,!\ \Phi_\delta^{(n)} = \sum_\beta h_\beta \prod_{r=1}^{k} X_{\beta_r}^{(n)};$$

where β *defines the partition* $\beta_1, \beta_2, \ldots \beta_k$ *of* δ, h_β *is the number of members of the corresponding class of conjugate operations of the symmetric group of degree* δ, *and*

$$X_\beta^{(n)} \begin{vmatrix} 1 & x^{2\beta} & y^{2\beta} \\ 1 & x^\beta & y^\beta \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x^{(n+2)\beta} & y^{(n+2)\beta} \\ 1 & x^\beta & y^\beta \\ 1 & 1 & 1 \end{vmatrix}.$$

When $n = 0$,

$$X_\beta^{(0)} = 1, \qquad \Phi_\delta^{(0)} = 1,$$

and the theorem is a truism.

We assume it true for all ternary quantics of order less than n.

Consider a gradient $\prod_{r=1}^{\delta} A_{\lambda_r, \mu_r}$, the sum of the suffixes of each coefficient must be equal to or less than n. Let each gradient be divided into two factors, let the first factor contain all the coefficients the sum of whose suffixes is n, and the second all the coefficients the sum of whose suffices in less than n. Let P_r be the generating function for those gradients in which r of the coefficients have the sum of the coefficients equal to n. Then

$$P_r = Q_r \cdot \Phi_{\delta-r}{}^{(n-1)},$$

where Q_r is obtained at once from the generating function for the gradients of degree r of the binary n-*ic*, *i.e.*,

$$Q_r = \frac{(x^{n+1}-y^{n+1})(x^{n+2}-y^{n+2})\dots(x^{n+r}-y^{n+r})}{(x-y)(x^2-y^2)\dots(x^r-y^r)} = \sum_\gamma \frac{h_\gamma^{(r)}}{r\,!}\prod_{s=1}^{l}\frac{x^{(n+1)\gamma_s}-y^{(n+1)\gamma_s}}{x^{\gamma_s}-y^{\gamma}}$$
$$= \sum_\gamma \frac{h_\gamma^{(r)}}{r\,!}\prod_{s=1}^{l}(X_{\gamma_s}{}^{(n)} - X_{\gamma_s}{}^{(n-1)}),$$

by Theorem XV. Here $h_\gamma^{(r)}$ is the number of members of the γ conjugate set in the symmetric group of degree r.

Thus we obtain

$$\Phi_\delta{}^{(n)} = \sum_{r=0}^{\delta} P_r = \sum_{r=0}^{\delta}\left[\sum_\gamma \frac{h_\gamma^{(r)}}{r\,!}\prod_{s=1}^{l}(X_{\gamma_s}{}^{(n)} - X_{\gamma_s}{}^{(n-1)})\right]\left[\sum_{\gamma'}\frac{h_{\gamma'}{}^{(\delta-r)}}{(\delta-r)!}\prod_{s=1}^{l}X_{\gamma'_s}{}^{(n-1)}\right].$$

The generating function for binary forms can be treated in exactly the same way, and the same equation is obtained, where now

$$X_\beta{}^n = \frac{[(n+1)\,\beta]}{[\beta]}\,;$$

by § 21 the result is

$$\sum_\beta h_\beta \prod_{r=1}^{k} X_{\rho_r}{}^{(n)}\,.$$

It is evident that there can be no linear relation with constant coefficients between the products $\prod_{r=1}^{k} X_{\rho_r}{}^{(n)}$; and hence the result is due to the values of the coefficients $h_\gamma^{(r)}$, $h_{\gamma'}^{(\delta-r)}$, and is quite independent of the form assigned to the functions $X_\beta{}^{(n)}$; hence for ternary forms

$$\Phi_\delta{}^{(n)} = \sum_{r=0}^{\delta} h_\beta \prod_{r=1}^{k} X_{\beta_r}{}^{(n)}$$

as was to be proved.

§ 38. It will be noticed that the generating function just obtained is the result of putting for every cycle of degree r in the symmetric group of degree δ the function $X_r{}^{(n)}$ and then dividing by $\delta\,!$. The same methods give us the generating function for gradient types of degree δ, and of any particular substitutional form. The result is

Theorem XXII—*The generating function* $\Phi_\alpha{}^{(n)}$ *for gradient types, of degree* δ *of ternary* n-*ics, of substitutional form* T_α *is*

$$\delta\,!\;\Phi_\alpha{}^{(n)} = \sum_\beta h_\beta\,\chi_\beta{}^{(\alpha)}\prod_{r=1}^{k} X_{\beta_r}.$$

These results may be extended at once to quaternary and higher forms. For quaternary forms we take

$$X_\beta \begin{vmatrix} 1 & x^{3\beta} & y^{3\beta} & z^{3\beta} \\ 1 & x^{2\beta} & y^{2\beta} & z^{2\beta} \\ 1 & x^{\beta} & y^{\beta} & z^{\beta} \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x^{(n+3)\beta} & y^{(n+3)\beta} & z^{(n+3)\beta} \\ 1 & x^{2\beta} & y^{2\beta} & z^{2\beta} \\ 1 & x^{\beta} & y^{\beta} & z^{\beta} \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

§ 39. As a verification of the results we may deduce from them the generating functions for types and compare them with those obtained in Q.S.A., VII,* Section VIII. Thus for ternary forms the generating functions for gradient types of the ternary *n-ic* is

$$\sum_\alpha f_\alpha \, \Phi_\alpha^{(n)} = \{X_1^{(n)}\}^\delta$$

the same result as that obtained by other means in § 44 of the paper just quoted.

Summary

The particular application, that is the main object of this paper, is the determination of a generating function for concomitants of ternary forms, of a nature more amenable to calculation than that obtained in a previous paper entitled "Some Generating Functions."† This object is achieved and a generating function for concomitants of any particular substitutional form, of quantics of finite or of infinite order, and with any number of variables is obtained. Amongst other results certain polynominal functions which appeared in the generating functions for binary forms in the paper just quoted have properties of interest which are discussed ; and some of these properties are extended to the corresponding functions with two or more variables. A one to one correspondence between the generating function for binary perpetuants of particular substitutional form and the perpetuants themselves is obtained by means of the tableaux and this leads at once to the corresponding extension of Grace's Theorem on irreducible perpetuant types.

Incidentally, certain properties of the Characteristic Function of Schur are considered, and a curious identity is obtained involving a series of determinants.

* Young, 'Proc. London Math. Soc.,' vol. 36, p. 304 (1934).

† 'Proc. London Math. Soc.,' vol. 35, p. 425 (1933).